A lattice approach to the conformal OSp($2S+2|2S$) supercoset sigma model Part I: Algebraic structures in the spin chain. The Brauer algebra.

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Abstract

We define and study a lattice model which we argue is in the universality class of the OSp($2S+2|2S$) supercoset sigma model for a large range of values of the coupling constant g_σ^2. In this first paper, we analyze in details the symmetries of this lattice model, in particular the decomposition of the space of the quantum spin chain $V^{\otimes L}$ as a bimodule over OSp($2S+2|2S$) and its commutant, the Brauer algebra $B_L(2)$. It turns out that $V^{\otimes L}$ is a nonsemisimple module for both OSp($2S+2|2S$) and $B_L(2)$. The results are used in the companion paper to elucidate the structure of the (boundary) conformal field theory.

1 Introduction

The solution of the $AdS_5 \times S^5$ worldsheet string theory is one of the cornerstones of the AdS/CFT duality program. Despite continuous effort and progress on classical aspects in particular [1], and the generally accepted presence of both integrability and conformal invariance symmetries, most aspects of the quantum theory remain elusive.

It is natural to try to understand some aspects of this quantum theory by first tackling simpler models with similar properties. The so called OSp($2S+2|2S$) coset model - specifically, a sigma model on the supersphere OSp($2S+2|2S$)/ OSp($2S+1|2S$) - is a very attractive candidate for such an exercise: like the $AdS_5 \times S^5$ worldsheet theory it is conformal invariant and its target space is a supergroup coset. Of course, it lacks other aspects such as the BRST structure of the string theory.

Apart from the string theory motivation, models such as the OSp($2S+2|2S$) coset model are extremely interesting from the pure conformal field theory point of view. Indeed, they are sigma models which are massless without any kind of topological term, and for a large range of values of the coupling constant g_σ^2. To make things more precise let us briefly remind the reader of some generalities. Supersphere sigma models have target super space the supersphere $S^{R-1,2S} :=$ OSp($R|2S$)/ OSp($R-1|2S$) and can be viewed as a "supersymmetric" extension of the nonlinear O(N) sigma models (which differs of course from the usual O(N) "supersymmetric" models). Use as coordinates a real scalar field

$$\phi := (\phi^1, \ldots, \phi^{R+2S})$$

where the first R components are bosons, the last $2S$ ones fermions, and the invariant bilinear form

$$\phi.\phi' = \sum J_{ij}\phi^i\phi'^j$$

where J is the orthosymplectic metric

$$J = \begin{pmatrix} I_R & 0 & 0 \\ 0 & 0 & -I_S \\ 0 & I_S & 0 \end{pmatrix}$$

I denoting the identity. The unit supersphere is defined by the constraint

$$\phi.\phi = 1$$

The action of the sigma model (conventions are that the Boltzmann weight is e^{-S}) reads

$$S = \frac{1}{2g_\sigma^2}\int d^2x\, \partial_\mu\phi.\partial_\mu\phi$$

The perturbative β function depends only on $R-2S$ to all orders (see, e.g., Ref [2]), and is the same as the one of the O(N) model with $N := R-2S$. Physics can be reliably understood from the first order beta function

$$\beta(g_\sigma^2) = (R-2S-2)g_\sigma^4 + O(g_\sigma^6)$$

The model for g_σ^2 positive flows to strong coupling for $R-2S > 2$. Like in the ordinary sigma models case, the symmetry is restored at large length scales, and the field theory is massive. For $R-2S < 2$ meanwhile, the model flows to weak coupling, and the symmetry is spontaneously broken. One expects this scenario to work for g_σ^2 small enough, and the corresponding Goldstone phase to be separated from a non perturbative strong coupling phase by a critical point.

The case we are interested in here is $R-2S = 2$, where the β function vanishes to all orders in perturbation theory, and the model is expected to be conformal invariant, at least for g_σ^2 small enough, the Goldstone phase being replaced by a phase with continuously varying exponents not unlike the low temperature Kosterlitz Thouless phase. How the group symmetry combines with the (logarithmic) conformal symmetry in such models is largely unknown. It is an essential question to be solved before any serious attempts to understanding universality classes in non interacting disordered 2D electronic systems can be contemplated [3].

The OSp($2S+2|2S$) coset model was considered in particular in two papers by Mann and Polchinski using the massless scattering and Bethe ansatz approaches. This is indeed a natural idea, since supersphere sigma models are in general integrable, and, when massive (ie $R-2S > 2$) can be described by a scattering theory involving particles in the fundamental representation of the group. The S matrix is well known

$$\check{S}(\theta) = \sigma_1(\theta)E + \sigma_2(\theta)P + \sigma_3(\theta)I$$

Here, I is the identity, P is the graded permutation operator, and E is proportional to the projector on the identity representation. For R, S arbitrary, factorizability requires that

$$\begin{aligned}
\sigma_1(\theta) &= -\frac{2i\pi}{(N-2)(i\pi-\theta)}\sigma_2(\theta) \\
\sigma_3(\theta) &= -\frac{2i\pi}{(N-2)\theta}\sigma_2(\theta)
\end{aligned}$$

where $N = R-2S$, while σ_2 itself is determined, up to CDD factors, by crossing symmetry and unitarity. One immediately observes that when $N = 2$, the amplitude σ_2 cancels out, leaving a scattering matrix with a simpler tensorial structure, since the P operator disappears. This corresponds to a particular point [4] on the sigma model critical line (where, among other things, the symmetry is enhanced to SU($2S+2|2S$)), the rest of which is not directly accessible by this construction.[1] The idea used in [5] is to consider an analytical continuation to R, S real, and an approach to $R-2S = 2$ with proper scaling of the mass. Though interesting results were obtained, the emphasis in these papers was not on conformal properties.

Another line of attack, more suited to the conformal aspects, was launched by Read and Saleur in 2001 [4], who proposed to use a lattice regularization to control the integrable features of the model. They obtained in this way the spectrum of critical exponents for several related sigma models on super target spaces, including the OSp($2S+2|2S$) coset one at a particular (critical) value of the coupling g_σ^2. The results exhibited several mysterious features, including a pattern of large degeneracies, and a set of values of the exponents covering (modulo integers) all the rationals. In two subsequent papers [6, 7], it was argued further that many algebraic properties of the conformal field theory could be obtained at the lattice level already. These include fusion, and the structure of conformal "towers" (see below for further details).

The work we present in this paper and its companion is an attempt at understanding the conformal field theoretic description of the OSp($2S+2|2S$) model for all values of the coupling by using a lattice regularization. Foremost in the lattice approach is the understanding of the algebraic structure of the lattice model - the algebra defined by the local transfer matrices and its commutant. While in the cases discussed in [7] most necessary results were already available in the mathematical literature, the situation here is much more complicated: in a few words, we have to deal, instead of the Temperley Lieb algebra, with the *Brauer algebra* whose representation theory, in the non semi-simple case, is far from fully understood. An important part of our work has consisted in filling up the necessary gaps of the literature, sometimes rigorously, but sometimes at the price of some

[1] The case $R = 2, S = 0$ is special and allows for an extension of the S matrix to the whole O(2) critical line.

conjectures. This algebraic work is the subject of the first paper, which we realize might be a bit hard to read for a physics reader. We capitalize on the algebraic effort in the companion paper, where the boundary conformal field theory for the coset sigma model is analyzed thoroughly.

In the second section of this paper we discuss generalities about lattice regularizations of O(N) sigma models in 2 dimensions and define the model we shall be interested in. In section 3, the transfer matrix, the loop reformulation and the associated Brauer algebra are introduced and discussed. Section 4 is the main section, where the full decomposition of the Hilbert space of the lattice model under the action of OSp($2S+2|2S$) and $B_L(2)$ is obtained. Our main result can be found in eqs. (4.36) and (4.37). Section 5 discusses aspects of the hamiltonian limit and section 6 contains conclusions. Technical aspects of representation theory are discussed further in the appendices.

For the reader's convenience, we provide here a list of notations used throughout the paper:

- osp($R|2S$) is the Lie superalgebra of the supergroup OSp($R|2S$)
- $B_L(N)$ is the Brauer algebra on L strings with fugacity for loops $N = R - 2S$
- $V_{R|2S} = \mathbb{C}^{R|2S}$ is the mod 2 graded vector space $\mathbb{C}^R \oplus \mathbb{C}^{2S}$ with even part $V_0 = \mathbb{C}^R$ and odd part $V_1 = \mathbb{C}^{2S}$. We shall often drop the indices $R, 2S$ in $V_{R|2S}$.
- $V^{\otimes L}$ is considered as a left osp($R|2S$) and right $B_L(N)$ bimodule
- $\lambda \vdash L$ stands for "λ is a partition of L" and λ' is the partition λ transposed
- Sym(L) and $\mathbb{C}$ Sym(L) are the symmetric group on L objects and its group algebra
- $T_L(q)$ is the Temperley Lieb algebra with fugacity for loops $q + q^{-1}$
- d and D are generic elements of $B_L(N)$ and OSp($R|2S$)
- $X_L = \{\mu \vdash L - 2k \mid k = 0, \ldots, [L/2]\}$ is the set of partitions labeling the weights of $B_L(N)$. $X_L(S) \subset X_L$ selects those of them which do realize on $V^{\otimes L}$
- Associate weights λ, λ^* are labels of OSp($R|2S$) irreps which are nonequivalent(identical) and become(split into two) isomorphic(nonequivalent) irreps under the restriction to the proper subgroup OSp$^+$($R|2S$) of supermatrices with sdet $D = +1$
- $H_L(S) = \{\lambda \in X_L(S) \mid \lambda_{r+1} \leq S\}$ is the set of hook shape partitions labeling the weights of osp($R|2S$) irreps appearing in $V^{\otimes L}$ and $Y_L(S) = H_L(S) \cup H_L(S)^*$
- $\Delta_L(\mu)$ are standard or generically irreducible representations of $B_L(N)$
- $S(\lambda)$, $g(\lambda)$, $G(\mu)$, $B_L(\mu)$, $D_L(j)$ are irreducible representations of $\mathbb{C}$ Sym(L), osp($R|2S$), OSp($R|2S$), $B_L(N)$ and $T_L(q)$ respectively.
- $\mathcal{I}g(\lambda), \mathcal{I}G(\mu), \mathcal{I}B_L(\mu)$ are direct summands of $V^{\otimes L}$ as a osp($R|2S$), OSp($R|2S$) and $B_L(N)$ module
- sc_λ are the supersymmetric generalization of O(N) symmetric functions
- $\chi'_\mu(d), \chi_\mu(d), \mathrm{sch}_\lambda(D)$ are characters of $\Delta_L(\mu)$, $B_L(\mu)$ and $G(\lambda)$.

2 The OSp($R|2S$) lattice models: generalities

2.1 The models and their loop reformulations

Lattice discretizations of O(N) sigma models have had a long history. The simplest way to go is obviously to introduce spins taking values in the target manifold - the sphere O(N)/ O($N-1$) - on the sites of a discrete lattice, with an interaction energy of the Heisenberg type $E = -J\sum_{<ij>} \vec{S}_i.\vec{S}_j$ (where . stands for the bilinear O(N)-invariant quadratic form). This is however difficult to study technically, as the number of degrees of freedom on each site is infinite. A possible way to go is to discretize the target space, leading to various types of

"cubic models" [8]. Another way which has proved especially fruitful in two dimensions has been to reformulate the problem of calculating the partition or correlation functions geometrically by using the techniques of high or low temperature expansions, thus obtaining graphs with complicated interaction rules and weights determined by properties of the underlying groups. The simplest of these formulations appeared in [9] where the authors studied the O(N) model on the honeycomb lattice in two dimensions, and replaced moreover the term $\prod_{<ij>} e^{\beta J \vec{S}_i . \vec{S}_j}$ by its considerably simpler high temperature approximation $\prod_{<ij>}(1 + K\vec{S}_i.\vec{S}_j)$, $K = \beta J$ [10]. Expanding the brackets, in say the calculation of the partition function, one can draw graphs by putting a bond between neighboring sites i and j whenever the term $\vec{S}_i.\vec{S}_j$ is picked up. The integral over spin variables leaves only loops, with a fugacity equal to N as there are N colors one can contract. Note that because of the very low coordination number of the honeycomb lattice, only self-avoiding loops are obtained. This leads to the well known self-avoiding loop gas partition function:

$$Z_{SAL} = \sum_{\mathcal{G}} K^E N^L \tag{2.1}$$

where the sum is taken over all configurations $\mathcal{G}$ of self avoiding, mutually avoiding closed loops in number L, covering a total of E edges. Note that once an expression such as (2.1) is written down, it is possible to analytically continue the definition of the model for N an arbitrary real number. Barring the use of superalgebras, only N integer greater or equal to one has a well defined meaning as a spin model (the case $N = 1$ coincides with the Ising model [2]). In two dimensions, the Mermin Wagner theorem prevents spontaneous symmetry breaking, so for N integer, critical behavior can only occur for $N = 1, 2$. Analysis of the same beta function suggests however that lattice models defined by suitable analytic continuation should have a Goldstone low temperature phase for all $N < 2$, though it says nothing about whether this phase might end by a second or first order phase transition.

Model (2.1) lacks interaction terms which would appear with less drastic choices of the lattice and the interactions: these are the terms where the loops intersect, either by going over the same edge, or over the same vertex, maybe many times. It has often been argued that such terms are irrelevant for the study of the critical points of the O(N) models in two dimensions. Most of the interest has focused on such critical points for $N \in [-2, 2]$, which have geometrical applications - in particular the case $N = 0$ is related with the physics of self-avoiding walks. It turns out however that intersection terms are crucial for the understanding of low temperature phases. Indeed, the model (2.1) *does have* a sort of Goldstone phase for $N \in [-2, 2]$ called the dense phase, but its properties are not generic, and destroyed by the introduction of a small amount of intersections. A simple way to see that the dense phase is not generic is that the exponents at $N = 2$ are always those of the Kosterlitz Thouless transition point: model (2.1) does not allow one to enter the low temperature phase of the XY model. Also, model (2.1) has a first order transition for $N < -2$, which is not the behavior expected from the sigma model analysis.

It was suggested in [11] that model (2.1) can be repaired by allowing for some intersections. The minimal scenario one can imagine is to define a similar model on the square lattice, and allow for self intersections at vertices only, so either none, one or two loops go through the same vertex. The resulting objects are often called trails. This gives the new partition function

$$Z_T = \sum_{\mathcal{G}'} K^E N^L w^I$$

where the sum is taken over all configurations $\mathcal{G}'$ of closed loops, which visit edges of the lattice at most once, in number L, covering a total of E edges, with I intersections.

The phase diagram of this model has not been entirely investigated. It is expected that at least for w small enough, the critical behavior obtained with $K = K_c$, $w = 0$ is not changed (though K_c is), while the low temperature behavior $K > K_c$ will be.

In [11] a yet slightly different version was considered corresponding, roughly, to the limit of very large K, where all the edges of the lattice are covered. The partition function of this fully packed trails model then depends on only two parameters:

$$Z_{FPT} = \sum_{\mathcal{G}''} N^L w^I \tag{2.2}$$

[2]The group O(2) is different from SO(2) $\simeq$ U(1) because of the additional $\mathbb{Z}_2$ freedom in choosing the sign of the determinant.

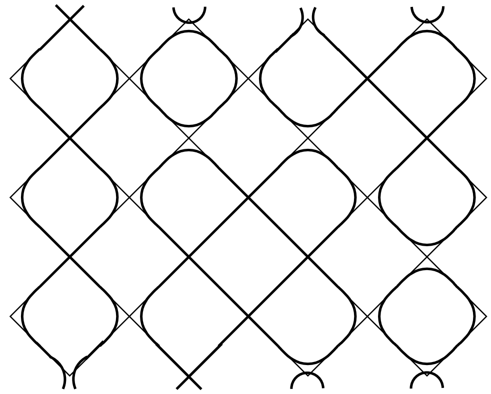


Figure 1: Dense intersecting loop covering of a lattice with annulus boundary conditions. Illustration of bulk (B), contractible (C), even (E) and odd (O) loops. Periodic imaginary time runs vertically

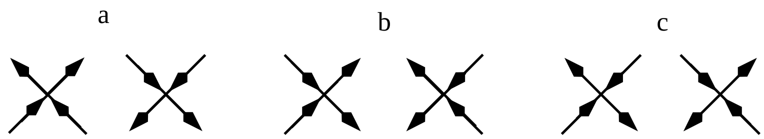


Figure 2: Vertices of the $\mathbb{Z}_2$ symmetric six vertex model

and an example of allowed configuration is given in figure 2.1. Numerical and analytical arguments suggest strongly that this model has all the generic properties of the O(N) sigma model in the spontaneously broken symmetry phase (such that one might derive from analytic continuation in N of the RG equations), for all $N \leq 2$.

Note that expression (2.2) can be obtained very naturally if instead of putting the degrees of freedom on the vertices, one puts them on the edges of the lattice. In this case, the minimal form of interaction involves two edges crossing at one vertex. Invariance under the O(N) group allows for three invariant tensors as illustrated on the figure 8, while isotropy and invariance under an overall scale change of the Boltzmann weights leaves one with a single free parameter, the crossing weight w. Graphical representation of the contractions on the invariant tensors reproduces eq. (2.2), as will be discussed below.

For $N < 2$, model (2.2) flows to weak coupling in the IR, and therefore it is expected that the critical properties of the corresponding low temperature (Goldstone) phase do not depend on w, a fact checked numerically in [11]. The case $N = 2$ is expected to be different: as mentioned already in the introduction, the beta function of the corresponding sigma model is exactly zero so the coupling constant does not renormalize. It is indeed easy to see that the loop model (2.2) with $N = 2$ is equivalent to the 6 vertex model with $a = b = 1 + w$, $c = 1$. Consider the vertices of the 6 vertex model as represented on figure 2. We chose isotropic weights $a = b$, c. We can decide to split the vertices of a configuration into pieces of oriented loops as represented on figure 3. For each vertex, there are two possible splittings, and we assume that they are chosen with equal probability. The loops obtained by connecting all the pieces together provide a dense covering of the lattice, and come with two possible orientations, hence a fugacity of two once the orientations are summed over. The loops can intersect, with a weight w given from the obvious correspondence:

$$\frac{a}{c} = \frac{b}{c} = \frac{1+w}{2}$$

and thus

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab} = 1 - \frac{2}{(1+w)^2}$$

We note that there are indeed *three* invariant tensors for the case of O(2). The corresponding projectors are E, $\frac{1}{2}(I - P)$, $\frac{1}{2}(I - E + P)$. They project respectively on two one dimensional representations, and on a single two dimensional one.

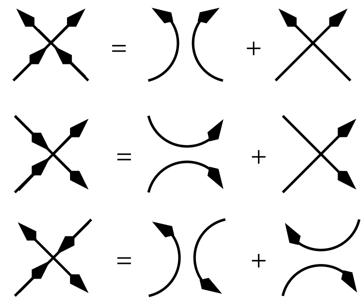


Figure 3: The mapping of the six vertex model onto the oriented loops

The parameter Δ covers the interval $[-1, 1]$ as $w \in [0, \infty]$. Changing Δ is well known to change the exponents of 6 vertex model, and therefore eq. (2.2) for $N = 2$ exhibits a critical line, which is in fact in the universality class of the continuous XY (O(2)) model in the low temperature (Kosterlitz Thouless) phase.

All what was said so far can be easily generalized to the case of spins taking values on a supersphere $\mathrm{OSp}(R|2S)/\mathrm{OSp}(R-1|2S)$. The fugacity of loops is now equal to $R - 2S$: this combination is the number of bosonic minus the number of fermionic coordinates, and follows from the usual fact that when contracting fermions along a loop, a minus sign is generated [3], see sec. 3.3. The loop model formulation therefore provides a convenient graphical representation of the discrete supersphere sigma models for all $R - 2S$, in particular $R - 2S \leq 2$ where interesting physics is expected to occur. This physics was explored in [11], and the expected results were obtained for $R - 2S < 2$. The purpose of this paper is to explore the more challenging $R - 2S = 2$ case.

Of course, at the naive level of partition functions and without worrying about boundary conditions, it looks as if there is no difference between the O(N) spin model and its supersphere cousins provided $R - 2S = N$. The point is that the *observables* of the models are different or, at the very least, come with different multiplicities. Indeed, consider for instance correlation functions of spin variables. In the O(2) case, the spin has only two components S^1, S^2, so one cannot build a totally antisymmetric tensor on three indices. This means that the corresponding operator (which has a nice geometrical interpretation to be given in the next paper) will not be present in this case, though it will be in the $\mathrm{OSp}(2S+2|2S)$ model when $S > 0$. Note that in general, correlators involving spins within the first R bosonic and the first $2S$ fermionic labels will be the same for *any* choice of group $\mathrm{OSp}(R'|2S')$ with $R' - 2S' = R - 2S$ and $R' \geq R$.[4] This is immediately proved by performing a graphical expansion of the correlator: variables outside of the set of the first R bosonic and the first $2S$ fermionic labels are not getting contracted with the spins in the correlators, and cancel against each other in the loop contractions.

A standard trick to extract the full operator content of a model is to study the partition function with different boundary conditions. Consider for instance the spin model on an annulus with some symmetry preserving boundary conditions in the space direction. With what we will call periodic boundary conditions (corresponding to taking the supertrace of the evolution operator) in the time direction, representations of $\mathrm{OSp}(2S+2|2S)$ will always be counted with their superdimension, and the partition function will be identical with the one of the O(2) case. But if we take antiperiodic boundary conditions, we will get a modified partition function (in the sense of [4]) which is a *trace* over the Hilbert space instead of a supertrace, counts all observables with the multiplicities (not supermultiplicities), and will turn out to be a very complex object.

A good algebraic understanding of the lattice model will be essential to make further progress, and, since the area is largely unexplored, this will occupy us for most of the rest of this first paper.

[3] The generalization of results for O(N) models to the case of orthosymplectic groups dates back to the work of Parisi and Sourlas [12].

[4] Provided, of course, that the boundary conditions imposed on the R' bosonic and $2S'$ fermionic degrees of freedom are the same in both cases.

3 Transfer matrices and algebra

3.1 Transfer matrices

As discussed briefly in the introduction, the $\mathrm{OSp}(R|2S)$ spin model we consider is most easily defined on a square lattice with degrees of freedom (states) on the edges and interactions taking place at vertices. The set of states on every edge is a copy of the base space V of the fundamental $\mathrm{OSp}(R|2S)$ representation. Interactions at a vertex can be encoded in a local transfer matrix t acting on $V^{\otimes 2}$ and commuting with the $\mathrm{OSp}(R|2S)$ supergroup action. We call t an intertwiner and write $t \in \mathrm{End}_{\mathrm{OSp}(R|2S)} V^{\otimes 2}$.

The Boltzmann weights of the model are components of the transfer matrix along a basis of intertwiners. A natural choice of basis are the projectors onto $\mathrm{OSp}(R|2S)$ irreducible representations appearing in the decomposition of the tensor product of two fundamental $\mathrm{OSp}(R|2S)$ representations. To find them one can apply the same (anti)symmetrization and trace substraction techniques used for reducing $\mathrm{O}(N)$ tensor representations. If $e_1, \ldots, e_{R+2S}$ is a mod 2 graded set of basis vectors in V with grading g, the decomposition of $V^{\otimes 2}$ will read

$$\begin{aligned} e_i \otimes e_j = & \frac{1}{2}\left(e_i \otimes e_j + (-1)^{\mathrm{g}(i)\,\mathrm{g}(j)} e_j \otimes e_i - \frac{2J_{ij}}{R-2S} \sum_{k,l} J^{kl} e_k \otimes e_l \right) \\ & + \frac{1}{2}\left(e_i \otimes e_j - (-1)^{\mathrm{g}(i)\,\mathrm{g}(j)} e_j \otimes e_i \right) + \frac{J_{ij}}{R-2S} \sum_{k,l} J^{kl} e_k \otimes e_l . \end{aligned} \tag{3.1}$$

Here J_{ij} is the $\mathrm{OSp}(R|2S)$ invariant tensor, $J^{ij} = (J^{-1})_{ij}$, and $g(i) = 1$ (resp. $g(i) = 0$) if i is fermionic (resp. bosonic). Each of the three terms on the l.h.s. of (3.1) transforms according to an irreps of $\mathrm{OSp}(R|2S)$, or, in other words, belongs to a simple $\mathrm{OSp}(R|2S)$ module.

Introduce the identity I, the graded permutation operator P (also known as braid operator), and E the Temperley Lieb operator (proportional to the projector on the trivial representation),

$$I_{ij}^{\;kl} = \delta_i^k \delta_j^l, \quad P_{ij}^{\;kl} = (-1)^{\mathrm{g}(i)\,\mathrm{g}(j)} \delta_i^k \delta_j^l, \quad E_{ij}^{\;kl} = J^{kl} J_{ij}. \tag{3.2}$$

In terms of projectors onto irreducible $\mathrm{OSp}(R|2S)$ modules, eq. (3.1) may be written in a more elegant way as

$$I = \frac{1}{2}\left(I + P - \frac{2}{R-2S} E \right) + \frac{1}{2}(I - P) + \frac{1}{R-2S} E .$$

Let P denote as usual the inversion of space, T the inversion of time and C the charge conjugation with the matrix J. One can check directly from definition (3.2) that P is C_{12}, P and T_{12} invariant, while E is P_{12} and $\mathsf{C}_{12}\mathsf{T}_{12}$ invariant. Moreover, E and I transform into each other under the $\pi/2$ rotation of the lattice R, while E and P are related by the crossing symmetry $\mathsf{C}_1\mathsf{T}_1$

$$\begin{aligned} \mathsf{R}: \; E_{ij}^{\;kl} &\longrightarrow J_{jj'} E_{k'i}^{\;\;lj'} J^{k'k} = I_{ij}^{\;kl} \\ \mathsf{C}_1\mathsf{T}_1: \; E_{ij}^{\;kl} &\longrightarrow J_{ii'} E_{k'j}^{\;\;i'l} J^{k'k} = P_{ij}^{\;kl}. \end{aligned}$$

Take I, E and P as basis of intertwiners in $\mathrm{End}_{\mathrm{OSp}(R|2S)} V^{\otimes 2}$. The local transfer matrix generally depends on three independent weights w_I, w_E and w_P. However, on a homogeneous and isotropic lattice one can normalize $w_I = w_E = 1$ and leave only the weight $w = w_P$. Finally, the local transfer matrix takes the form

$$t(w) = I + wP + E. \tag{3.3}$$

On a diagonal lattice with open boundaries represented in fig. 4 choose the time in vertical direction. The notation of sites at a fixed time is such that the left edge i and right edge $i+1$ meet at vertex i. Let $t_i(w) \in \mathrm{End}_{\mathrm{OSp}(R|2S)} V^{\otimes L}$ denote a transfer matrix acting nontrivially only at vertex i according to eq. (3.3). From the figure it is clear that odd and even times are inequivalent. The transfer matrix T, propagating one step forward at equivalent times, may be written as a product $T = YX$ of one layer transfer matrices

$$X = \prod_{i=1}^{[(L-1)/2]} t_{2i}, \quad Y = \prod_{i=1}^{[L/2]} t_{2i-1}, \tag{3.4}$$

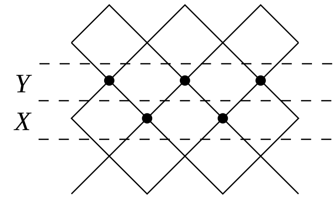


Figure 4: The one layer transfer matrices X and Y represented on a diagonal lattice of width 6.

schematically shown in fig. 4.

The simplest way to define a partition function that depends on the whole spectrum of the transfer matrix T is by taking the *trace* of T at a certain power β. Selecting other boundary conditions with some nontrivial symmetry generally amounts to restricting the whole space of states of the model to a subspace compatible with the symmetry of chosen boundary conditions. What exactly we mean by "symmetry of boundary conditions" will be explained later in sec. 3.3. For the moment let us just say that it is convenient to consider a more general class of boundary conditions, called quasiperiodic, in which T^β is "twisted" by the action of an element D of the supergroup. Define the quasiperiodic partition function to be

$$Z_D = \mathrm{str}_{V^{\otimes L}}\, D^{\otimes L} T^\beta. \tag{3.5}$$

We must take the *supertrace* in eq. (3.5) if we want the quasiperiodic partition function to be well defined. For instance, when $D = J^2$ we get the usual trace partition function and when D equals to the identity matrix we get the supertrace partition function.

Note that because D is a supermatrix, the tensor product in $D^{\otimes L}$ has to be graded, that is

$$D^{\otimes 2} \cdot \eta \otimes \xi = D \cdot \eta \otimes D \cdot \xi \quad \Rightarrow \quad \left(D^{\otimes 2}\right)^{ij}_{\ \ kl} = (-1)^{\mathrm{g}(k)(\mathrm{g}(j)+\mathrm{g}(l))} D^i_{\ k} D^j_{\ l}$$

After inserting the local transfer matrix from eq. (3.3) in eq. (3.4) and expanding the transfer matrix T, the quasiperiodic partition function reads as a sum of weighted products of E_i's and P_i's. Such linearly independent products must be considered as words of a *transfer matrix algebra*, while intertwiners E_i and P_i are *generators* of this algebra. In the next section we identify this algebra as a representation of the Brauer algebra.

3.2 The Brauer algebra

For an abstract introduction to the Brauer algebra see ref. [13, 14] while in the context of osp($R|2S$) centralizer algebra see ref. [15]. We collect in this section some well known facts about the Brauer algebra we shall use in the next sections.

Let E_i and P_i, $i = 1, \ldots, L$ act nontrivially as E and P in eq. (3.2) only at the sites $V_i \otimes V_{i+1}$ of $V^{\otimes L}$. One can check that for P_i and E_i so defined the following relations hold:

$$\begin{gathered} P_i^2 = 1, \quad E_i^2 = N E_i, \quad E_i P_i = P_i E_i = E_i, \\ P_i P_j = P_j P_i, \quad E_i E_j = E_j E_i, \quad E_i P_j = E_j P_i, \\ P_i P_{i\pm1} P_i = P_{i\pm1} P_i P_{i\pm1}, \quad E_i E_{i\pm1} E_i = E_i, \\ P_i E_{i\pm1} E_i = P_{i\pm1} E_i, \quad E_i E_{i\pm1} P_i = E_i P_{i\pm1}. \end{gathered} \tag{3.6}$$

In the second line of these relations i and j are supposed to be nonadjacent sites.

Relations (3.6) (is one of the many ways to) define the $B_L(N)$ Brauer algebra (also denoted sometimes by the names of braid-monoid algebra or degenerate Birman-Wenzel-Murakami algebra [16, 17, 18]). Note that this algebra depends on a single, generally complex, parameter N, and contains the maybe more familiar Temperley Lieb algebra, generated by E_i's alone, and the symmetric group algebra, generated by P_i alone.

For N fixed and L big enough, the OSp($R|2S$) spin models provide highly unfaithful representations of the Brauer algebra $B_L(N)$. This is because, in $V^{\otimes L}$, the generators P_i and E_i satisfy additional higher order relations $\mathcal{R}$ on top of (3.6).[5] For a simple example, consider the O(2) spin model on a lattice of width 3. The projector

[5]This situation is similar to what happens for models with SL(N) symmetry in the fundamental representation, and corresponding quotients of the Hecke algebra.

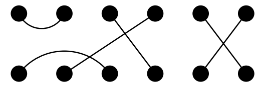


Figure 5: The graphical representation of the word $P_5 P_3 E_1 P_2$ in $B_6(N)$.

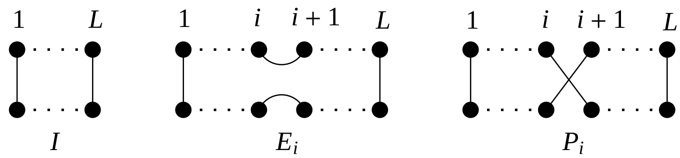


Figure 6: Graphical representation of generators I, E_i and P_i.

onto the antisymmetric tensor of rank 3 is zero, thus, $\mathcal{R}$ contains the additional relation $1 + P_1P_2 + P_2P_1 = P_1 + P_2 + P_1P_2P_1$. Our spin models in general provide representations of the quotient algebras $B_L(N)/\mathcal{R}$. The set of relations $\mathcal{R}$ can be explicitly described for $S = 0$, see [19, 20] and references therein, and we have little to say about the case $S > 0$.

The first step in understanding the spectrum of the transfer matrix T brings up the question of $B_L(N)$ irreducible representations, and of their multiplicities in T for a particular choice of R and S. This leads us to discussing some results about the representation theory of the Brauer algebra.

The most natural representation to begin with is the *adjoint representation*. It admits a diagrammatic representation in terms of graphs on $2L$ points in which every vertex has degree 1. Usually one orders the $2L$ points on two horizontal parallel lines as shown in fig. 5. Let B_L denote the vector space spanned on the $(2L-1)!!$ such diagrams.

The product $d_1 * d_2$ of two diagrams d_1 and d_2 is performed by putting d_1 on top of d_2 and replacing each of the loops in the resulting diagram with N. Define diagrammatically the identity I and the generators E_i, P_i as represented in fig. 6. One can check that the graphical representation of generators with the multiplication $*$ of diagrams satisfy all of the eqs. (3.6). The left action of generators on B_L via the multiplication $*$ of diagrams provide the adjoint representation of $B_L(N)$.

From the graphical representation we see that B_L has a series of invariant subspaces $B_L = B_L^L \supset B_L^{L-2} \supset \cdots \supset B_L^\tau$, where B_L^m is spanned on diagrams with fewer than m vertical lines and $\tau = L \mod 2$. The vector space spanned on diagrams with exactly m vertical lines may be defined as a $B_L(N)$ module by the coset $B_L'^m = B_L^m / B_L^{m-2}$. The left action of $B_L(N)$ on this modules may be seen as a modified multiplication $*_m$ of diagrams

$$d_1 *_m d_2 = \begin{cases} d_1 * d_2, & \text{if it has } m \text{ vertical lines} \\ 0, & \text{otherwise} \end{cases}$$

Under the left action of the algebra the position of horizontal lines in the bottom of a diagram does not change. For a given configuration of the horizontal lines in the top of a diagram and a given pattern of intersections of vertical lines there are $(L-m-1)!!C_L^m$ possibilities of choosing the configuration of horizontal lines in the bottom of the diagram. This simply means that $B_L'^m$ decomposes into a direct sum of $(L-m-1)!!C_L^m$ equivalent modules. The coset representative $B_L''^m$ of these equivalent left modules is spanned on $m!(L-m-1)!!C_L^m$ graphs on L points with every vertex having degree 0 or 1 and a labeling with numbers $1, \ldots, m$ of free vertices. An example of such a labeled graph is shown in fig. 7. If the labellings are omitted the resulting graph is called a partial diagram.

The labeling of the m free points of a labeled graph is a permutation π in the symmetric group $\mathrm{Sym}(m)$. The labeled graphs will provide a representation of the Brauer algebra, which is irreducible for generic values of N, if we take the labellings π in an irreducible representation of $\mathrm{Sym}(m)$. We call such representations generically irreducible. Let μ be a partition of m, which we write as $\mu \vdash m$. In a more algebraic language the definition of generically irreducible left modules translates to

$$\Delta_L(\mu) := B_L''^m \otimes_{\mathrm{Sym}(m)} S(\mu), \quad \mu \vdash m, \tag{3.7}$$

where $S(\mu)$ is an irreducible $\mathrm{Sym}(m)$ module. In view of later numerical analysis we give below a basis in $\Delta_L(\mu)$ and describe the action of $B_L(N)$ on this basis.

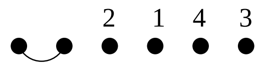


Figure 7: The top of the diagram represented in fig. 5. The labeling permutation is $(2, 1, 4, 3)$.

Let $p \otimes \pi$ denote the labeling of a partial diagram p with the permutation π, $v_1, \ldots, v_{f_\mu}$ be a set of basis vectors in $S(\mu)$ and $\rho_\mu(\sigma)$ be the matrix of the permutation σ in the representation ρ_μ. A natural basis in $\Delta_L(\mu)$ is given by all pairs $p \otimes v_i$. The action of a diagram $d \in B_L(N)$ on a basis vector is

$$d \cdot p \otimes v_i = \begin{cases} d *_m p \otimes \rho_\mu(\sigma^{-1}) v_i, & \text{if } d * p \text{ has } m \text{ free points} \\ 0, & \text{otherwise,} \end{cases} \tag{3.8}$$

where σ is the labeling of $d * g$ and g is the partial diagram p labeled with the identity permutation. The dimensions d_μ of $\Delta_L(\mu)$ is $f_\mu (L - m - 1)!! C_L^m$.

In simple words, a generically irreducible module is a span on graphs on L points, obtained by choosing m points among L, pairing all the others (this gives the multiplicity $(L - m - 1)!!$ since intersections are allowed), choosing for the m unpaired ones a representation of the permutation group and setting to zero the action of any Brauer diagram that reduces the number m of unpaired points.

The generically irreducible representations labeled by $\mu \vdash L - 2k$, $k = 0, \ldots, [L/2]$ appear in the decomposition of the adjoint representation with multiplicity given by their dimension d_μ when $B_L(N)$ is semisimple.

Let us conclude with a few words about the reducibility of generically irreducible modules $\Delta_L(\mu)$. For integer N and a number of strings $L > N$ the Brauer algebra is not semisimple and, as a consequence, certain of the modules $\Delta_L(\mu)$ become reducible, though they remain *indecomposable*.[6] The irreducible components appearing in such reducible modules $\Delta_L(\mu)$ are far from being understood (the situation is much worse than in the case of the nonsemisimple Temperley Lieb algebra [21, 22]). Numerical computations based on the diagonalization of the transfer matrix in the diagrammatic representation of $B_L(N)$ restricted to $\Delta_L(\mu)$ decreases in efficiency very fast with increasing L, compared to the ideal case where the transfer matrix is restricted to an irreps of $B_L(N)$. This is because for big L and μ fixed the number of irreducible components in $\Delta_L(\mu)$ “goes wild” and there are a lot of “accidental degeneracies” in the spectrum of the transfer matrix restricted to $\Delta_L(\mu)$.

However, a significant progress in this direction has been recently made in [23, 24]. Let us note that, as described in [23], the content of (at least some) $\Delta_L(\mu)$’s can be computed by repeated applications of Frobenius reciprocity applied to the short exact sequence of [14] describing the structure of the induced modules $B_{L+1}(N) \otimes_{B_L(N)} \Delta_L(\mu)$.

In the end we recall the basic results for the Temperley Lieb algebra, to allow a quick comparison with Brauer. Temperley Lieb algebra diagrams are a subset of Brauer algebra diagrams subject to the constraint that no intersections between edges are allowed. The dimension of the algebra is given by the Catalan numbers $(2L)!/L!(L + 1)!$. The main line of reasoning for finding generically irreducible modules follows the same way, except there is no available action of the symmetric group on vertical lines. Therefore, the analogue of the labeled graphs will be the partial diagrams, in which no free points may be trapped inside an edge. The number of such graphs is $C_{L-1}^n - C_{L-1}^{n-2}$, where $n = (L - m)/2$ is the number of edges. The generically irreducible modules $D_L(m)$ are parametrized by the number $m = L, L - 2, \ldots$ of free points in the graphs.

The presented facts about the Brauer algebra should be enough to understand the loop gas reformulation of $\mathrm{OSp}(R|2S)$ spin model, which we give in the next section.

3.3 Loop reformulation of $\mathrm{OSp}(R|2S)$ spin models: the algebraic point of view

The emergence of dense intersecting loops becomes transparent if we take the local transfer matrices in the adjoint representation of the Brauer algebra. This simply amounts to replacing in eq. (3.3) the generators I, E_i and P_i defined by eq. (3.2) with the diagrams in fig. 6. The adjoint local transfer matrix is represented in fig. 8.

We now define a loop model on a diagonal lattice represented in fig. 4, with reflecting boundaries on the left and right (ie, free boundary conditions in the space direction) and identified boundaries on the top and

[6] We adopt here the physicist’s habit of calling indecomposable a module which is reducible though not fully reducible. Therefore the set of indecomposables does not contain the irreducibles, unlike in most of the math literature.

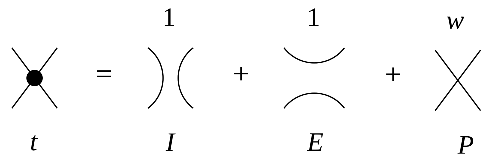


Figure 8: Possible vertex of interactions in the loop model.

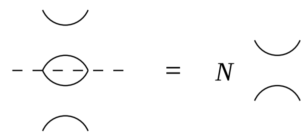


Figure 9: Emergence of bulk loops.

bottom (quasiperiodic boundary conditions in the time direction.) The states of this model are coverings of the lattice with dense *intersecting* loops. Dense means that every edge on the lattice necessarily belongs to a loop. Avoiding loop vertices have weight 1 and intersections come with weight w. There are two possible ways for a line to close in a loop. The first one comes from the graphical representation of the relation $E^2 = NE$ in fig. 9. We call such loops *bulk* loops. Clearly the fugacity of bulk loops is fixed to N by the Brauer algebra. The second possibility is that the ends of the line close in the identified points of the top and bottom boundaries of the lattice. We call such loops *cycles*. The boundary condition we consider have an annulus geometry and, thus, a cycle can be either *contractible* or *uncontractible*. The fugacity of cycles is not fixed by the algebra. In fact, as we explain below, this is exactly the degree of freedom allowing for multiple mappings from the $\mathrm{OSp}(R|2S)$ spin models with $R - 2S = N$ fixed and the dense intersecting loop model with fugacity N for loops.

We start by evaluating the trace $\mathrm{tr}_{V^{\otimes L}}\, d$ of a diagram d in the spin representation and then we generalize the result for quasiperiodic boundary conditions given by $\mathrm{str}_{V^{\otimes L}}\, D^{\otimes L} d$. We follow the same line of reasoning as in [13].

A cycle in a diagram d is the subgraph on the set of points belonging to a loop if we identify its top and bottom vertices. By an abuse of language we call the corresponding loop also cycle. If we put a diagram d_1 to the left of a diagram d_2 we get a new diagram which we denote $d_1 \otimes d_2$. Let $c_1, \ldots, c_l$ be the cycles in d. We can separate them by permuting the top and bottom vertices of d with the same permutation π

$$\pi * d * \pi^{-1} = c_1 \otimes \cdots \otimes c_l.$$

Thus the trace of a diagram depends only on the weights of cycles

$$\mathrm{tr}_{V^{\otimes L}}\, d = \mathrm{tr}\, c_1 \ldots \mathrm{tr}\, c_l.$$

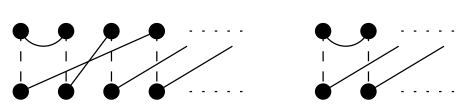
More than that, the weight of a cycle depends only on how many times it winds the annulus.

Indeed, if a cycle on $2m$ points has no horizontal lines, then, by applying the same permutation to the top and bottom vertices we can bring it to the cycle $P_1 \ldots P_{m-1}$. This is because permutations with one cycle are conjugate in $\mathrm{Sym}(m)$.

If a cycle c has a horizontal edge between the first and the second vertex in the top then it has the same weight as a certain cycle c' on four points less then c

$$\mathrm{tr}\, c = \frac{1}{N}\,\mathrm{tr}\, E_1 * c = \frac{1}{N}\,\mathrm{tr}\, c * E_1 = \frac{1}{N}\,\mathrm{tr}\, E_1 \otimes c' = \mathrm{tr}\, c'. \tag{3.9}$$

If we compare the c on the left with c' on the right it is clear that, in the end of the iterative application of



eq. (3.9), the final cycle can be interpreted as being the initial cycle c maximally contracted on the annulus.

In the end, the only weights we need to compute explicitly are that of the cycles E and $P_1 \dots P_{m-1}$

$$\begin{aligned}
\mathrm{tr}_{V^{\otimes 2}} E &= J_{i_1 i_2} J^{i_1 i_2} = N \\
\mathrm{tr}_{V^{\otimes m}} P_1 \dots P_{m-1} &= (-1)^{\mathrm{g}(i_1)(\mathrm{g}(i_2)+\dots+\mathrm{g}(i_m))} \delta_{i_1=\dots=i_m} = R + (-1)^{m+1} 2S.
\end{aligned}$$

For boundary conditions twisted by the matrix $D \in \mathrm{OSp}(R|2S)$ the generalized weights are computed to be

$$\begin{aligned}
\mathrm{str}_{V^{\otimes 2}} D^{\otimes 2} E &= N \\
\mathrm{str}_{V^{\otimes m}} D^{\otimes m} P_1 \dots P_{m-1} &= \mathrm{str}\, D^m.
\end{aligned} \tag{3.10}$$

In the fundamental representation, every supermatrix D is diagonalizable. The diagonal form of $D \in \mathrm{OSp}^+(R|2S)$ in the fundamental representation is determined by exponentiating elementary weights ϵ_i and δ_j introduced in sec. A.1. Thus, D restricted to V_0 has eigenvalues $x_1, x_1^{-1}, \dots, x_r, x_r^{-1}, (x_{r+1} = 1)$ and restricted to V_1 has eigenvalues $y_1, y_1^{-1}, \dots, y_S, y_S^{-1}$. The braces in (x_{r+1}) mean that x_{r+1} appears for odd R only. Eq. (3.10) can now be rewritten

$$\mathrm{str}\, D^m = \sum_{i=1}^{r} (x_i^m + x_i^{-m}) + (1) - \sum_{j=1}^{S} (y_j^m + y_j^{-m}). \tag{3.11}$$

For $D \in \mathrm{OSp}^-(R|2S)$ only the eigenvalues in V_0 change with respect to the previous case. There are of the form $x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_r, x_r^{-1}, x_{r+1} = -1$ for R odd, while for R even $x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_{r-1}, x_{r-1}^{-1}$ and $x_r = 1, x_r' = -1$. Instead of eq. (3.11) one has now

$$\begin{aligned}
\mathrm{str}\, D^m &= \sum_{i=1}^{r} (x_i^m + x_i^{-m}) + (-1)^m - \sum_{j=1}^{S} (y_j^m + y_j^{-m}), && R \text{ odd} \\
\mathrm{str}\, D^m &= \sum_{i=1}^{r-1} (x_i^m + x_i^{-m}) + 1 + (-1)^m - \sum_{j=1}^{S} (y_j^m + y_j^{-m}), && R \text{ even.}
\end{aligned}$$

To summarize the basic results in this section, let $\mathcal{G}$ be a dense loop covering of the lattice, I be the number of intersections, B be the number of bulk loops, C be the number of contractible loops (cycles) and $E(O)$ be the number of loops winding the annulus an even(odd) number of times.

On the annulus the trace partition function (which would correspond to antiperiodic boundary conditions in the (imaginary) time direction) of the $\mathrm{OSp}(R|2S)$ spin model may be reformulated as a dense intersecting loop model in the following way

$$Z = \sum_{\mathcal{G}} w^I N^{B+C+E} (R + 2S)^O. \tag{3.12}$$

We see that it does depend on R, S separately and not only on N.

Meanwhile the supertrace partition function (which would correspond to periodic couplings) reads

$$Z = \sum_{\mathcal{G}} w^I N^L,$$

where $L = B + C + E + O$ is the total number of "loops" and, since it depends on N only, is the same as for the $\mathrm{O}(N)$ model. One can say that taking the supertrace in the partition function is equivalent to restricting the $\mathrm{OSp}(R|2S)$ supersymmetry of the spin model to a, smaller, $\mathrm{O}(N)$ symmetry.

Denote the spectrum of the transfer matrix of the $\mathrm{OSp}(R|2S)$ spin model by $\Sigma_S(N)$. We have the following important inclusion property

$$\Sigma_0(N) \subset \Sigma_1(N) \subset \Sigma_2(N) \subset \dots. \tag{3.13}$$

The only difference between the $\mathrm{OSp}(R|2S)$ and $\mathrm{OSp}(R-2|2S-2)$ quasiperiodic partition functions is the weight of uncontractible cycles. For $D \in \mathrm{OSp}(R|2S)$ a matrix with eigenvalues $x_i = 1$ and $y_j = -1$, except $y_1 = 1$, the weight of uncontractible cycles is, according to eq. (3.11) either N or $R + 2S - 4$. Notice that these are exactly the weights of uncontractible cycles in the trace partition function for the $\mathrm{OSp}(R-2|2S-2)$ spin model, which proves eq. (3.13).

We will use the inclusion property (3.13) in the next section to derive some information about the indecomposable representations of $\mathrm{OSp}(R|2S)$ appearing in the decomposition of $V^{\otimes L}$.

4 Decomposition of $V^{\otimes L}$

4.1 General results

Assume the transfer matrix be a generic element of $B_L(N)$. The action of $B_L(N)$ on the tensor space $V^{\otimes L}$ was defined in the beginning of sec. 3.2. The complete picture of the reducibility of the transfer matrix can be conveniently encoded in the decomposition of $V^{\otimes L}$ into a direct sum of $B_L(N)$ indecomposable modules

$$V^{\otimes L} \simeq \bigoplus_{\lambda \in Y_L(S)} m_\lambda \mathcal{I}B_L(\lambda), \tag{4.1}$$

where m_λ denotes the multiplicity of isomorphic indecomposable $B_L(N)$ modules $\mathcal{I}B_L(\lambda)$ (it does not depend on L), and the set $Y_L(S)$ is defined implicitly by the formula, and will be defined explicitly below. We remind the reader that $V^{\otimes L}$ is not necessarily a semisimple $B_L(N)$ module if $L > N$, so the modules $\mathcal{I}B_L(\lambda)$ appearing on the rhs of eq. (4.1) can be reducible.

The question of computing degeneracies of eigenvalues of the spin transfer matrix is easier to treat by looking at the centralizer $\mathsf{Z} := \mathrm{End}_{B_L(N)} V^{\otimes L}$, which acts on $V^{\otimes L}$ from the left if one consider $B_L(N)$ acting from the right. The dimension of indecomposable modules $\mathcal{I}G(\mu)$ in the decomposition of $V^{\otimes L}$ as a Z-module

$$V^{\otimes L} \simeq \bigoplus_{\mu \in X_L(S)} n_L^\mu \mathcal{I}G(\mu) \tag{4.2}$$

will give the desired degeneracies. This is due to the fact that n_L^μ are dimensions of simple $B_L(N)$ modules $B_L(\mu)$ appearing as constituents of $\mathcal{I}B_L(\lambda)$ in eq. (4.1), while m_λ are dimensions of simple G modules $G(\lambda)$ appearing as constituents of $\mathcal{I}G(\mu)$ in eq. (4.2). Taking the character of both eqs. (4.1,4.2) one can see that the number $b(\lambda,\mu)$ of irreducible components $B_L(\mu)$ in $\mathcal{I}B_L(\lambda)$ is equal to the number $g(\mu,\lambda)$ of irreducible components $G(\lambda)$ in $\mathcal{I}G(\mu)$.

Because the action of $\mathrm{osp}(R|2S)$ commutes with $B_L(N)$ we have that $\mathrm{osp}(R|2S) \subset \mathsf{Z}$. However, when $V^{\otimes L}$ is semisimple it follows from the Wedderburn decomposition theorem that $\mathsf{Z} \simeq \mathbb{Z}_2 \times \mathrm{osp}(R|2S)$. In the following we suppose that there is still a Schur duality between $\mathrm{osp}(R|2S)$ and the quotient of $B_L(N)$ faithfully represented on $V^{\otimes L}$. This allows us to give an algorithm to compute the lhs of eqs. (4.1,4.2) for small tensor powers L and get some intuition about the general structure of $\mathcal{I}B_L(\lambda)$ and $\mathcal{I}G(\mu)$.

The set of partitions $X_L = \{\mu \vdash L-2k \mid k = 0,\ldots,[L/2]\}$ labels $B_L(N)$ irreps, while $X_L(S) \subset X_L$ selects those of them which do realize on the tensor space $V^{\otimes L}$. Denote by $J(S) \subset B_L(N)$ the double sided ideal defined by $V^{\otimes L} \cdot J(S) = 0$. The annihilator $J(0)$ is diagrammatically described in [20]. Under the homomorphism $\rho : B_L(N) \to B_L(N)\,/J(S)$, the indecomposable modules $\Delta_L(\mu)$ give rise to induced modules $\delta_L(\mu) = \Delta_L(\mu)\,/J(S)\cdot\Delta_L(\mu)$. Clearly, $\delta_L(\mu)$ is a tensor representation and can be generated by trace substraction and symmetrization as

$$V^{\otimes L}\mathcal{T}_{L-2k}e_\mu E_{L-2k}\ldots E_{L-1}, \tag{4.3}$$

where $\mu \vdash L-2k$, $\mathcal{T}_{L-2k} \in B_L(N)$ extracts all the traces from the tensor space $V^{\otimes L-2k}$ and e_μ acts nontrivially only on $V^{\otimes L-2k}$ as a Young symmetrizer. The double sided ideal $J(S)$ is completely characterized by the set of weights $X_L(S) = \{\mu \in X_L \mid J(S)\cdot B_L(\mu) = 0\}$. Note that $X_L(S) \subset X_{L+2k}(S), k \geq 1$. The surviving indecomposable tensor modules $\delta_L(\mu)$ are given by $\Delta_L(\mu)$ with irreducible components $B_L(\nu), \nu \notin X_L(S)$ removed. The quotient $B_L(N)/J(S)$ can be carried out by imposing the vanishing of all words $W_\mu := \mathcal{T}_{L-2k}e_\mu \in B_L(N)$ with $\mu \in X_L/X_L(S)$. It is useful to notice that not all of these conditions are independent and as one can see from eq. (4.3) $W_\mu = 0 \Rightarrow W_\nu = 0$ if $\mu \subset \lambda$.

As discussed in sec. A.1 and A.2, the $\mathrm{osp}(R|2S)$ irreducible components of $\mathcal{I}G(\mu)$ are indexed (up to an equivalence under the action of the outer automorphism τ induced by the symmetry of the Dynkin diagram of $\mathrm{osp}(R|2S)$ when R even) by the set $H_L(S) = \{\lambda \in X_L(S) \mid \lambda_{r+1} \leq S\}$ of hook shape partitions. Representing the supergroup as a semidirect product $\mathrm{OSp}(R|2S) = \mathbb{Z}_2 \times \mathrm{OSp}^+(R|2S)$, the elements of $Y_L(S)$ naturally acquire the structure of a couple of the form $1\times\lambda$ or $\varepsilon\times\lambda$ if $\lambda_S < r$ and $\tau\times\lambda$ if $\lambda_S \geq r$, where $1,\varepsilon,\tau$ are the trivial, alternating (superdeterminant) and two dimensional representations of $\mathbb{Z}_2 = \mathrm{OSp}(R|2S)/\,\mathrm{OSp}^+(R|2S)$. Thus, every $\lambda \in H_L(S)$ gives rise to two $\mathrm{OSp}(R|2S)$ inequivalent irreps with highest weights $\lambda := 1\times\lambda$ and the associate $\lambda^* := \varepsilon\times\lambda$ if $\lambda_S < r$ and a single self-associate irreps of highest weight $\lambda = \lambda^* := \tau\times\lambda$ if $\lambda_S \geq r$. For typical $\lambda \in H_L(S)$, one can realize the $\mathrm{OSp}(R|2S)$ irreps λ,λ^* on tensors $T_L(\lambda), T_L(\lambda^*)$ and describe their

symmetry by some Young tableaux. As discussed in details in sec. B, the Young tableau corresponding to $T_L(\lambda^*)$ can be constructed by adding a border strip to the Young tableau of shape λ corresponding to $T_L(\lambda)$. Ultimately, this is justified by the fact that eq. (4.5,4.6) gives the right characters for $T_L(\lambda), T_L(\lambda^*)$ and that they coincide up to $\mathrm{sdet}\, D$. Although atypical representations cannot be realized as tensor representations we represent the associate weight λ^* of an atypical weight λ by a Young tableau such that λ^*/λ is a skew partition described in sec. B and sec. C.

The idea is to exploit the fact that the characters of indecomposable modules $\Delta_L(\mu)$, given in [13], do not depend on the semisimplicity of $B_L(N)$. This and some properties of generalized Schur functions, which are summarized in [15], can be used to prove that

$$\mathrm{str}_{V^{\otimes L}}\, D^{\otimes L} d = \sum_{\mu \in X_L} sc_\mu(D) \chi'_\mu(d), \tag{4.4}$$

is true even for all L. Here $\chi'_\mu(d)$ is the character of $d \in B_L(N)$ in the representation provided by $\Delta_L(\mu)$. The functions $sc_\mu(D)$ are polynomials in the eigenvalues of $D \in \mathrm{OSp}(R|2S)$, which where introduced for the first time by Bars in [25] in an early attempt to describe the supercharacters of $\mathrm{OSp}(R|2S)$. They can be defined recursively as

$$sc_n(D) = \oint \frac{dz}{2\pi i} \left(\frac{1}{z^{n+1}} - \frac{1}{z^{n-1}} \right) \frac{1}{\mathrm{sdet}(1 - zD)}, \tag{4.5}$$

and

$$sc_\mu(D) = \frac{1}{2} \det \Big(sc_{\mu_j - i - j + 2}(D) + sc_{\mu_j + i - j}(D) \Big). \tag{4.6}$$

For $L \leq N$ the Brauer algebra $B_L(N)$ is semisimple and $J = 0$. Consequently, $\Delta_L(\mu)$ are irreducible and $X_L = X_L(S)$. Because of the commuting actions of $\mathrm{OSp}(R|2S)$ and $B_L(N)$ one can naturally consider $V^{\otimes L}$ as a $\mathrm{OSp}(R|2S)$-$B_L(N)$-bimodule, with $\mathrm{OSp}(R|2S)$ acting from the left and $B_L(N)$ from the right. Then, eq. (4.4) can be understood as a consequence of the decomposition

$$V^{\otimes L} \simeq \bigoplus_{\mu \in X_L} G(\mu) \otimes \Delta_L(\mu), \quad L \leq N \tag{4.7}$$

with sc_μ being actual characters of tensor irreducible modules $G(\mu)$ as shown in [15].

For R, S such that $L > N$, the polynomials sc_μ cannot generally be interpreted as the character of some $\mathrm{OSp}(R|2S)$ representation. As we have seen in sec. 3.3, $\mathrm{str}_{V^{\otimes L}}\, D^{\otimes L} d$ can be brought to the form $N^h \prod_m \mathrm{str}_V\, D^m$ and eq. (4.4) is not more then a simple equality between two polynomials in eigenvalues of D. Moreover, the two eqs. (4.2,4.4) are still compatible, even if there are much more elements in X_L then in $X_L(S)$. This is possible because sc_μ are not functionally independent when $L > N$. Then, for $\mu \notin Y_L(S)$ the polynomials sc_μ can be written in terms of functionally independent sc_λ with $\lambda \in Y_L(S)$ by means of modification rules for characters

$$sc_\mu = \sum_{\lambda \in Y_L(S)} a(\mu, \lambda) sc_\lambda \tag{4.8}$$

given in [26] and discussed in details in sec. C.

The fundamental eq. (4.4) is useful for small widths L, when it is possible to compute the number $b'(\mu, \nu)$ of irreducible components $B_L(\nu)$ in $\Delta_L(\mu)$ either by repeated applications Frobenius reciprocity, as explained in [23], or by numerically diagonalizing the transfer matrix of sec. (3.1) in the adjoint representation of $B_L(N)$ and detecting the "accidental degeneracies" in its spectrum. Indeed, from the explicit definition (3.7) it is clear how to restrict the adjoint transfer matrix to indecomposable modules $\Delta_L(\mu)$. After we described in details the action of generators on the basis of $\Delta_L(\mu)$ in sec. 3.2, the algorithm of a numerical diagonalization is straightforward.

The information about the structure of $\Delta_L(\mu)$ and the modification rules in eq. (4.8) can now be used to bring eq. (4.4) to the form

$$\mathrm{str}_{V^{\otimes L}}\, D^{\otimes L} d = \sum_{\substack{\mu, \nu \in X_L \\ \lambda \in Y_L(S)}} a(\nu, \lambda) b'(\nu, \mu) sc_\lambda(D) \chi_\mu(d). \tag{4.9}$$

We see that $\mu \in X_L(S)$ iff [7] there is at least one $\lambda \in Y_L(S)$ such that $\sum_\nu a(\nu, \lambda) b(\nu, \mu) \neq 0$. To determine $g(\mu, \lambda)$ one has to decompose the factor of χ_μ in eq. (4.9) as a sum of $\mathrm{OSp}(R|2S)$ irreducible characters, which are

[7] Although sc_λ, $\lambda \in H_L(S)$ are not irreducible $\mathrm{osp}(R|2S)$ characters, one can still use them as a basis for representing the character of any representation.

explicitly known, as far as we know, only for OSp(3|2) and OSp(4|2). Given the huge order of the set of weights X_L, it may seem that calculations according to eq. (4.9) are extremely cumbersome already for small L. The simplifying point is that $a(\nu,\lambda)$ (or $b'(\nu,\mu)$) is non zero only if both weights are in the same equivalence class of $Y_L(S)$ (or $X_L(S)$). The splitting of $Y_L(S)$ (or $X_L(S)$) into equivalence classes, called *blocks* and described in details in sec. B, is with respect to an equivalence relation between irreducible components of indecomposable OSp$(R|2S)$ (or $B_L(N)$) modules.

An important consequence of the fact that OSp$(R|2S)$ supertrace partition functions depends only on the O(N) part of the spectrum is the vanishing of the superdimension $\mathrm{sdim}\,\mathcal{I}G(\mu) = 0$ for all indecomposable modules with $\mu \notin X_L(0)$. A more restrictive criterion for $\mathcal{I}G(\mu)$ supercharacters deriving from the full inclusion sequence in eq. (3.13) can be derived by taking a matrix D with eigenvalues $x_1 = y_1$ and $x_i \neq y_j$ for $i = 1\ldots,r$, $j = 1,\ldots,S$. Then, it can be seen from eqs. (3.11,3.12) or eqs. (4.5,4.6,4.4) that any OSp$(R|2S)$ quasiperiodic partition function will also be an OSp$(R-2|2S-2)$ quasiperiodic partition function. As a consequence $X_L(0) \subset X_L(1) \subset \cdots \subset X_L(S)$ and the supercharacters of $\mathcal{I}G(\mu)$ vanish when $\mu \in X_L(S)/X_L(S-1)$ and D can be embedded in OSp$(R-2|2S-2)$.

For $S=0$ the modules $V^{\otimes L}$ is semisimple. Therefore rad $B_L(L) \subset J(0)$. On the other hand, if S is big enough $J(S) = 0$. It could be interesting to understand the relation between the sequence $J(0) \supset J(1) \supset \cdots \supset J(S) = 0$ and the cohomology of the radical rad $B_L(N) \supset \mathrm{rad}^2 B_L(N) \supset \cdots \supset 0$.

Observe that a filtration similar to that of $X_L(S)$ is available on $Y_L(S)$ by the degree of atypicality of its elements. In fact, we explain in sec. B of the appendix how the weights in a block of $X_L(S)$ or $Y_L(S)$ can be organized by the number of *removable balanced continuous border strips* in the corresponding Young tableau. This number can be interpreted as the degree of atypicality when the corresponding partition represents an OSp$(R|2S)$ weight.

4.2 O(2) spin model

Let V be a two dimensional vector space endowed with an action of O$(2,\mathbb{R})$. The action of O(2) on the tensor space $V^{\otimes L}$ is

$$D \cdot V^{\otimes L} = \underbrace{DV \otimes \cdots \otimes DV}_{L}, \quad D \in \mathrm{O}(2).$$

$B_L(2)$ acts according to the following definition of generators E_i, P_i

$$E_i = \underbrace{1_2 \otimes \cdots \otimes 1_2}_{i-1} \otimes E \otimes 1_2 \otimes \cdots \otimes 1_2$$
$$P_i = \underbrace{1_2 \otimes \cdots \otimes 1_2}_{i-1} \otimes P \otimes 1_2 \otimes \cdots \otimes 1_2,$$

where P, E have the following representation on $V^{\otimes 2}$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.10}$$

(note that E differs in some essential way from the projection operator onto the singlet representation in the usual $SU(2)$ basis). The decomposition of $V^{\otimes L}$ as a O(2)-$B_L(2)$-bimodule is simply

$$V^{\otimes L} \simeq \bigoplus_{\mu \in X_L(0)} G(\mu) \otimes B_L(\mu). \tag{4.11}$$

Here

$$X_L(0) \text{ is composed of partitions } \mu_0 = \emptyset, \mu_{0^*} = 1^2, \text{ and } \mu_k = k, k \geq 1.$$

The tensor representations $G(\mu_k)$ are irreducible with dimensions $\dim G(\mu_k) = 1, k = 0, 0^*$ and $\dim G(\mu_k) = 2, k \geq 2$. The representation $G(\mu_0)$ is the trivial one and $G(\mu_{0^*})$ is the associate one dimensional $\det D$ representation. At the restriction to SO(2) $\simeq$ U(1) the representations $G(\mu_0)$ and $G(\mu_{0^*})$ become equivalent, while

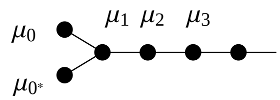


Figure 10: Edges in the graph represents nonzero elements of the fusion matrix C.

$G(\mu_k), k \geq 1$ splits into two nonequivalent one dimensional representations $e^{\pm ik\phi}$ where ϕ is the U(1) angle. The $B_L(2)$ representations $B_L(\mu)$ are irreducible as well and are constructed by acting with $B_L(2)$ on the tensor module in eq. (4.3). The dimensions of simple modules $B_L(\mu_k)$ are easily computed by looking at the first row of C^L, where C is the fusion matrix

$$G(\mu_k) \otimes V \simeq \bigoplus C_{k,l} G(\mu_l) \tag{4.12}$$

described by a D_L type Dynkin diagram with labeling of the nodes shown in fig. 10. It is not hard to solve the recurrence relations satisfied by $d(L,k) := \dim B_L(\mu_k)$

$$\begin{aligned} d(L+1,0) &= d(L+1,0^*) = d(L,1), \\ d(L+1,1) &= 2d(L,0) + d(L,2), \\ d(L+1,k) &= d(L,k+1) + d(L,k-1), \quad k \geq 2 \end{aligned}$$

and get that $d(L,k) = C_L^{[L/2]+k}$ except the case when L is even and $k = 0$ for which $d(L,0) = d(L,0^*) = C_L^{L/2}/2$. These are, as expected, the number of eigenvalues of the transfer matrix for the 6 vertex model in the sector of spin $s_z = k/2$.

We are interested in giving an algebraic description of the 6 vertex model transfer matrix algebra. In other words we want to identify the Brauer algebra annihilator $J := J(0)$ of $V^{\otimes L}$ and carry out the quotient $B_L(2)/J$.

All the $B_L(2)$ weights $\lambda \in X_L$ such that $\mu_k \subset \lambda$ satisfy either $\nu_0 = 1^3 \subseteq \lambda$ or $\nu_1 = 21 \subseteq \lambda$. Thus, it is enough to consider $L = 3$ and impose the vanishing of the double sided ideal of the word $W_0, W_1 \in B_L(2)$ projecting onto $\Delta_3(\nu_i), i = 0, 1$. As explained in the beginning of the previous section $W_i = \mathcal{T}_3 e_{\nu_i}$, where $\mathcal{T}_3$ extracts all the traces from $V^{\otimes 3}$ and e_{ν_i} are the Young symmetrizers corresponding to ν_i.

The projector $\mathcal{T}_3$ can be found by looking at the form of an arbitrary tensor G_{ijk} after extracting all of its traces

$$G_{ijk} - \frac{\delta_{ij}}{4}\Big(3G_{\cdot\cdot k} - G_{k\cdot\cdot} - G_{\cdot k\cdot}\Big) - \frac{\delta_{ik}}{4}\Big(3G_{\cdot j\cdot} - G_{j\cdot\cdot} - G_{\cdot\cdot j}\Big) - \frac{\delta_{jk}}{4}\Big(3G_{i\cdot\cdot} - G_{\cdot i\cdot} - G_{\cdot\cdot i}\Big),$$

which gives

$$\mathcal{T}_3 = 1 - \frac{1}{4}\Big(3E_1 - E_2E_1 - P_2E_1\Big) - \frac{1}{4}\Big(3P_1E_2P_1 - E_1P_2 - E_2P_1\Big) - \frac{1}{4}\Big(3E_2 - E_1E_2 - P_1E_2\Big)$$

and clearly $\mathcal{T}_3E_1 = \mathcal{T}_3E_2 = \mathcal{T}_3P_1E_2P_1 = 0$.

The Young symmetrizer e_{ν_0} is

$$e_{\nu_0} = \frac{1}{6}\Big(1 + P_1P_2 + P_2P_1 - P_1 - P_2 - P_1P_2P_1\Big)$$

and $e_{\nu_1} = e_{T_1} + e_{T_2}$, where

$$\begin{aligned} e_{T_1} &= \frac{1}{3}\Big(1 - P_1P_2P_1\Big)\Big(1 + P_1\Big) \\ e_{T_2} &= \frac{1}{3}\Big(1 - P_1\Big)\Big(1 + P_1P_2P_1\Big) \end{aligned}$$

are the projectors onto the *standard* Young tableau $T_1 = [12,3]$ and $T_2 = [13,2]$. The two orthogonal projectors e_{T_1} and e_{T_2} are independent only if we restrict to the right $B_L(2)$ action. In fact, the left ideal of the word $W_1 = 0$ is the same as the double sided ideal of the word $\mathcal{T}_3 e_{T_1} = 0$.

The condition $W_0 = 0$ gives the following restriction

$$1 + P_1P_2 + P_2P_1 = P_1 + P_2 + P_1P_2P_1 \tag{4.13}$$

on generators P_1, P_2. Putting $P_i = 1 - Q_i, i = 1, 2$ one can see that eq. (4.13) implies that Q_i are Temperley Lieb operators with $Q_i^2 = 2Q_i$. There are no more restrictions that can be drawn from the conditions $W_0 = 0$, because W_0 is a one dimensional projector.

Before exploring the next vanishing condition let us revise the the defining relations of $B_L(2)$ given in eq. (3.6)

$$E_iP_i = P_iE_i = E_i \quad \Rightarrow \quad Q_iE_i = E_iQ_i = 0 \tag{4.14}$$
$$P_iP_{i+1}E_i = E_{i+1}E_i \quad \Rightarrow \quad Q_iQ_{i+1}E_i = E_{i+1}E_i + Q_{i+1}E_i - E_i \tag{4.15}$$
$$E_{i+1}P_iP_{i+1} = E_{i+1}E_i \quad \Rightarrow \quad E_{i+1}Q_iQ_{i+1} = E_{i+1}E_i + E_{i+1}Q_i - E_{i+1}, \tag{4.16}$$

which imply

$$E_iQ_{i\pm1}E_i = E_i \tag{4.17}$$
$$Q_iQ_{i+1}E_i = Q_iE_{i+1}E_i \tag{4.18}$$

Observe that although the algebra has now two Temperley Lieb operators their role is not symmetric yet at this stage.

Next, the condition $\mathcal{T}_3 e_{T_1} = 0$ implies

$$1 + P_1 - P_1P_2P_1 - P_1P_2 = 2E_1 + E_2 - E_1E_2 - 2E_2E_1 - E_1P_2 + E_2P_1$$

which after inserting $P_i = 1 - Q_i$ with the help of eqs. (4.15,4.18) becomes

$$Q_1 + 2Q_2 - Q_2Q_1 - 2Q_1Q_2 = E_1 + 2E_2 - 2E_2E_1 - E_1E_2 + E_1Q_2 - E_2Q_1. \tag{4.19}$$

Multiplying eq. (4.19) by Q_2 on the right we get:

$$E_2Q_1Q_2 = E_2E_1 + E_2Q_1 - E_2 = E_1Q_2 + Q_1Q_2 - Q_2 = E_2E_1Q_2. \tag{4.20}$$

which can be used to rewrite eq. (4.19) as

$$Q_1Q_2 + Q_2Q_1 - Q_1 - Q_2 = E_1E_2 + E_2E_1 - E_2 - E_1. \tag{4.21}$$

Multiplying by E_i, Q_i on the left and on the right of eq. (4.21) and using only the relations between Q_i, the relations between E_i and eq. (4.14) one can get all the eqs. (4.15–4.20) and also

$$Q_1E_2E_1 = Q_1Q_2 + Q_1E_2 - Q_1 = E_2E_1 + Q_2E_1 - E_1 = Q_1Q_2E_1$$
$$Q_1E_2Q_1 = Q_1$$

which establish a complete symmetry between E_i and Q_i.

The double sided ideal of $\mathcal{T}_3 e_{T_1} = 0$ is composed of four linearly independent words — two generated by the left action and other two generated by the right action of $B_L(2)$. It is useful to note that after taking the quotient of $B_3(2)$ we are left with 10 independent words instead of 15, which is exactly what we need for the 6 vertex local transfer matrix.

We give the following abstract definition to the 6 vertex model transfer matrix algebra $\mathcal{V}_L := \mathrm{End}_{\mathrm{O}(2)} V^{\otimes L}$ in term of generators E_i, Q_i

$$E_i^2 = 2E_i, \quad E_iE_{i\pm1}E_i = E_i, \quad Q_i^2 = 2Q_i, \quad Q_iQ_{i\pm1}Q_i = Q_i \tag{4.22}$$
$$E_iQ_i = Q_iE_i = 0 \tag{4.23}$$
$$Q_iQ_{i+1} + Q_{i+1}Q_i - Q_i - Q_{i+1} = E_iE_{i+1} + E_{i+1}E_i - E_i - E_{i+1} \tag{4.24}$$
$$E_iE_j = E_jE_i, \quad Q_iQ_j = Q_jQ_i, \quad E_iQ_j = Q_jE_i$$
$$|i - j| > 1, \quad i, j = 1, \ldots, L - 1$$

The defining relations are symmetric under the transposition T, which changes the multiplication order, under the reflection $R : E_i \to E_{L-i}$ and under the involution $E^* = Q$. Thus, if $W = 0$ then $W^T = 0$, $W^R = 0$ and $W^* = 0$ is also true for any word $W \in \mathcal{V}_L$.

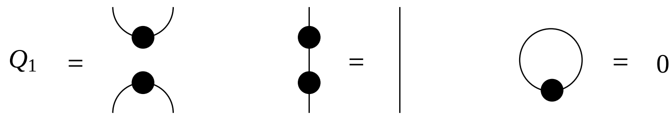


Figure 11: The generator Q_1 is represented as E_1 with its horizontal edges marked by a blobbed. The conditions satisfied by the blob are represented on the right.

Introducing the operators $S_i = 1 - E_i - Q_i$, with the property $S_i^2 = 1$, one can rewrite eq. (4.20) as

$$Q_{i+1} = S_i E_{i+1} S_i.$$

Thus, one can eliminate all of the generators $Q_i, i \geq 2$ and leave only Q_1 subject to satisfy

$$\begin{aligned} Q_1 E_1 = E_1 Q_1 = 0, &\quad Q_1^2 = 2Q_1 \\ Q_1 E_2 Q_1 = Q_1, &\quad E_2 Q_1 E_2 = E_2 \\ Q_1 E_j = E_j Q_1, &\quad j \geq 3. \end{aligned} \tag{4.25}$$

Denote by d_L the extension of the ordinary Temperley Lieb algebra, generated by E_i, with the additional generator Q_1 satisfying eqs. (4.25). We see that d_L and $\mathcal{V}_L$ are isomorphic algebras. The graphical interpretation for the reduced words (products of generators of minimum length) of d_L and its relation to the blob algebra and the Temperley Lieb algebra of type D is discussed [27]. The generators E_i are diagrammatically represented as usual, whereas Q_1 is represented as E_1 with each of its horizontal edges marked by an *involutive* blob as shown in fig. 11. An unblobbed loop is identified with 2, while a blobbed loop with 0. Thus, we see that d_L is a subalgebra of the blob algebra composed of all planar diagrams on $2L$ points with an *even* number of blobbed edges. The dimension of d_L is, as explained in [27], half the dimension of the blob algebra, that is $C_{2L}^{L}/2$.

There are several important consequences arising from the isomorphism between $\mathcal{V}_L$ and d_L from the point of view of integrability. First of all, we check that indeed the solution to the Yang-Baxter equation

$$R_1(u)R_2(u+v)R_1(v) = R_2(v)R_1(u+v)R_2(u) \tag{4.26}$$

provided by the algebra $\mathcal{V}_3$ coincides with the well known *XXZ* spin chain *R*-matrix.

For that, consider the ansatz $R(u) = I + f(u)Q + g(u)E$ and plug it in eq. (4.26). Choosing as basis set in $\mathcal{V}_3$ the 10 words $1, E_1, E_2, E_1E_2, E_2E_1, Q_1E_2, E_2Q_1, Q_1E_2E_1, E_1E_2Q_1$ we get two independent functional equations

$$E_1 : \; F(f,g) - F(g,f) = g'f - gf' + fg'' - f''g \tag{4.27}$$

$$E_2 : \; F(f,g) + F(g,f) = (f''g' + f'g'')(f+g), \tag{4.28}$$

where $F(f,g) = f'' + f' - f + f'f''(2 + f + g)$. The primed functions are evaluated in u, the unprimed in $u+v$ and the double primed in v. All other words provide the same third equation, which is a consequence of eqs. (4.27,4.28). The solution to the system of eqs. (4.27,4.28) is

$$f(u) = \frac{\sin\lambda - \sin u}{2\sin(\lambda - u)} - \frac{1}{2} \tag{4.29}$$

$$g(u) = \frac{\sin\lambda + \sin u}{2\sin(\lambda - u)} - \frac{1}{2}, \tag{4.30}$$

with an arbitrary constant λ. Taking Q and E in the representation provided by the eq. (4.10) we find the famous *XXZ* spin chain *R*-matrix

$$R_{XXZ}(u) = \begin{pmatrix} \sin(\lambda - u) & 0 & 0 & 0 \\ 0 & \sin\lambda & \sin u & 0 \\ 0 & \sin u & \sin\lambda & 0 \\ 0 & 0 & 0 & \sin(\lambda - u) \end{pmatrix}, \quad \Delta = -\cos\lambda \tag{4.31}$$

as expected.

Clearly, an integrable system in $\mathcal{V}_L$ has to be related to an integrable system in d_L because of the isomorphism of these two algebras. However, the ansatz $R(u) = 1 + g(u)E$ plugged into the eq. (4.26) gives only the isotropic point ($\Delta = \pm 1$) solution $g(u) = u/(1-u)$. The only possibility to give a richer content to the integrability in d_L is by introducing nontrivial boundary conditions. This means that the anisotropy of the *XXZ* spin chain can be generated by introducing nontrivial boundary conditions at the isotropic points, an observation made earlier from a slightly different perspective in [28].

4.3 OSp(4|2) spin model

The representation theory of the superalgebra osp(4|2) is summarized in [29]. As we have already mentioned, all of osp(4|2) irreducible characters have been computed and indecomposable representations classified. We give a brief reminder of these results in sec.A.3 and make some remarks, based on the general discussion in sec. A.2, on the difference between the representation theory of the supergroup OSp(4|2) and its Lie superalgebra.

The tensor space $V^{\otimes L}$, seen as a OSp(4|2) module, can be represented as a direct sum $V^{\otimes L} = V^{(0)} \oplus V^{(1)}$ of a part "lifted" from O(2)

$$V^{(0)} = \bigoplus_{\lambda \in Y_L(0)} n_L^\lambda G(\lambda) \tag{4.32}$$

and a projective part

$$V^{(1)} = \bigoplus_{\lambda \in Y_L(1)/Y_L(0)} n_L^\lambda \mathcal{P}G(\lambda), \tag{4.33}$$

where $\mathcal{P}G(\lambda)$ is the projective cover of $G(\lambda)$. This decomposition can be proved by induction on L using two facts:

- The tensor product between atypical irreducible representations with highest weights labeled by one row partitions (see bellow) and V decomposes to [8]

  $$G(k) \otimes V \simeq G(k+1) \oplus G(k1) \oplus G(k-1).$$

  This is proved by counting the dimensions on the right/left hand sides and, then, observing that $G(k1)$ is typical and $G(k \pm 1)$, being in different blocks, cannot give rise to indecomposables.

- The tensor product of a projective module with any other module is projective, thus, decomposing to a direct sum of projectives.

In the following we use the fundamental eqs. (4.4,4.9) to decompose $V^{\otimes L}$ as a $\mathsf{Z} = \mathrm{End}_{B_L(2)} V^{\otimes L}$ module and verify the assumption that $\mathsf{Z} = \mathbb{Z}_2 \times \mathrm{osp}(4|2)$ by comparing the result to eqs. (4.32,4.33).

The conditions of atypicality for a osp(4|2) weight λ are given in sec. A.3. In the partition notation we adopt, these are equivalent to

$$\lambda_1' = 1 \text{ or } \lambda_1 + 1 = \lambda_1' \text{ or } \lambda_2 = \lambda_1'.$$

Typical weights satisfy none of atypicality conditions listed above. Note that typical representations are irreducible, have vanishing superdimension, and are simultaneously projective and injective. This means they cannot be a constituents of any other osp(4|2) representations without being a direct summand. One can say they are "their own blocks".

The supercharacters of associate OSp(4|2) irreps λ, λ^* satisfy $\mathrm{sch}_{\lambda^*}(D) = \mathrm{sdet}\, D\, \mathrm{sch}_\lambda(D)$. For typical weights, the polynomials sc_λ give the right OSp(4|2) irreducible character. Because of the modification rules, see sec. C, it is possible to define a partition λ_{mod} such that $sc_{\lambda_{\mathrm{mod}}}(D) = \mathrm{sdet}(D) sc_\lambda(D)$. Therefore, it is convenient to identify the associate weight $\lambda^* = \varepsilon \times \lambda$ with the partition λ_{mod}. The Young tableau of λ^* can be constructed by replacing the orthogonal part of the Young tableau of λ by its associate, that is by putting $(\lambda^*)_2' = 4 - \lambda_2'$ and leaving all other columns unchanged. For instance $(1^4)^* = 2^4$. Exceptions are the typical weights λ such that $\lambda_1' < 4 - \lambda_2'$. The only such weights are $\lambda = 1^3, 21$ or $l1, l \geq 3$ and we put $(1^3)^* = 32^3, (21)^* = 3^2 21$ and $(l1)^* = l32, l \geq 3$.

[8] Modulo irreps labeled by $\lambda \notin Y_L(0)$, this decomposition provides the same fusion matrix as eq. 4.12. Thus, $Y_L(0)$ multiplicities in $V_{4|2}^{\otimes L}$ are the same as those in $V_{2|0}^{\otimes L}$. Rather then a coincidence, this is a direct manifestation of the algebra inclusion $\mathrm{End}_{\mathrm{OSp}(4|2)} V_{4|2}^{\otimes L} \supset \mathrm{End}_{\mathrm{O}(2)} V_{2|0}^{\otimes L}$ at the level of dimensions of irreps.

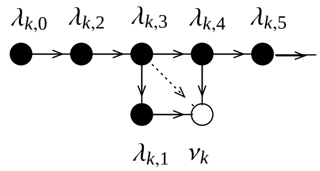


Figure 12: Two weights $\lambda_{k,l}$ and $\lambda_{k,l'\neq l}$ are connected by a continuous line iff $\lambda_{k,l} \subset \lambda_{k,l'}$ and there is no other weight between them. The weight $\lambda_{k,l}$ is connected to $\lambda_{k,l'}$ by a dotted arrow iff $\Delta_L(\lambda_{k,l})$ has an irreducible component $B(\lambda_{k,l'})$. Its multiplicity is always one.

The atypical OSp(4|2) weights can be labeled by two integers k and l, where k denotes the isomorphism class, also called block.

In the partition notation, the block $k = 0$ is composed of weights $\lambda_{0,0} = \emptyset, \lambda_{0,1} = (1^2)^* := 3^2 2^2$ and $\lambda_{0,l} = (l1^l)^* := l2^2 1^{l-1}, l \geq 2$. The associate block $k = 0^*$ is composed of weights $\lambda_{0^*,0} = 1^2$ and $\lambda_{0^*,1} = (\emptyset)^* := 3^4, \lambda_{0^*,l} = \lambda^*_{0,l} = l1^l, l \geq 2$.

The self associate blocks $k \geq 1$ are composed of weights $\lambda_{k,0} = k, \lambda_{k,1} = k^* := k3^2$ ($3^3 2$ for $k = 1$ and $3^3 1$ for $k = 2$), $\lambda_{k,l} = \lambda^*_{k,l} = kl1^{l-2}$ for $2 \leq l \leq k$ and $\lambda_{k,l} = \lambda^*_{k,l} = l(k+1)1^{l-1}$ for $l \geq k+1$.

With the given notation for associate weights one can check with the help of [29] the following decomposition of polynomials $sc_{\lambda_{k,l}}$ as a sum of OSp(4|2) supercharacters sch

$$\begin{aligned} &sc_{\lambda_{k,0}} = \mathrm{sch}_{\lambda_{k,0}},\ sc_{\lambda_{k,1}} = -\mathrm{sch}_{\lambda_{k,3}} + \mathrm{sch}_{\lambda_{k,1}},\ sc_{\lambda_{k,2}} = \mathrm{sch}_{\lambda_{k,2}} + \mathrm{sch}_{\lambda_{k,0}}, \\ &sc_{\lambda_{k,l}} = \mathrm{sch}_{\lambda_{k,l}} + \mathrm{sch}_{\lambda_{k,l-1}} + (-1)^{l-1}\,\mathrm{sch}_{\lambda_{k,1}},\ l \geq 2. \end{aligned} \tag{4.34}$$

This is done in two steps. First one show that eqs. (4.34) hold for a supermatrix D with sdet $D = 1$.[9] At this step is yet impossible to distinguish between associate representations. In order to do so, one has to explicitly construct the elements of the enveloping Lie superalgebra connecting the maximal vectors of irreducible components of indecomposable highest weight modules and, then, look at their symmetry under the outer automorphism τ. See sec. A.3 for details.

We have just listed all the elements of $Y_L(1)$. Eq. (4.34) is a bijection between sch_λ and sc_λ. As a consequence, OSp(4|2) and $B_L(2)$ weights can be labeled by the same set $Y_L(1) = X_L(1)$ in the partition notation we have adopted. This is supporting the assumption that there is some sort of exact equivalence between the category of OSp(4|2) and $B_L(2)$ modules on $V^{\otimes L}$. Bellow all the weights are partitions and, to avoid confusion, we write $\lambda \in Y_L(1)$ if λ is considered as a OSp(4|2) weight and $\lambda \in X_L(1)$ if it is considered as a $B_L(2)$ weight.

Let us show that the terms in eq. (4.4) with $\lambda \notin Y_L(1)$ do not actually contribute to $\mathrm{str}_{V^{\otimes L}} D^{\otimes L} d$. First note that if χ_λ cancels out from eq. (4.4) then certainly $\delta_L(\lambda)$ in eq. (4.3) is a trivial module. Therefore any module $\delta_L(\nu)$ will also be trivial if $\lambda \subset \nu$. Second, if χ_λ does not contribute to eq. (4.4) when $\lambda \vdash L$ then it does not contribute to it for any L. Thus, it is enough to prove for every k that the weights just greater (by inclusion) then $\lambda_{k,l}$ do not contribute to eq. (4.4) when the are allowed for the first time to appear.

Let $\lambda \in Y_L(1)$ be a typical (associate) weight. Then, as we show in sec. B, $\lambda \in X_L(1)$ is a minimal partition (with respect to the inclusion in its block). There will be a unique weight $\nu \notin Y_L(1)$ just greater then λ and, *a priori*, sc_ν can modify to $\pm sc_\lambda$. It is proved by induction in sec. C that a positive sign would imply atypicality conditions on λ and, thus, $sc_\nu = -sc_\lambda$. Moreover, from [23] we know that $\Delta_L(\lambda)$ has one composition factor $B_L(\nu)$. Taking $L = |\nu|$, we see that the contribution to eq. (4.4) of χ_ν from $\Delta_L(\lambda)$ cancels out with the one from $\Delta_L(\nu)$.

Before proceeding to nontrivial blocks we need to know the number of irreducible components $B_L(\lambda_{k,l'})$ in $\Delta_L(\lambda_{k,l})$. According to [23], the graph representing the partial ordering (by inclusion) of weights in a block k determines the *required information* about the content of modules $\Delta_L(\lambda_{k,l})$. The ordering graph is represented in fig. 12.

Now, let $\lambda_{k,l} \in Y_L(1)$ be an atypical (associate) weight. Then, any weight $\nu \notin X_L(1)$ such that $\lambda_{k,l} \subset \nu$ satisfies $\nu_k \subseteq \nu$, with ν_k represented by a white dot in fig. 12. The explicit form of ν_k is $\nu_0 = 432^2 1, \nu_{0^*} = 43^3 1, \nu_1 = 43^2 21, \nu_2 = 43^2 1^2, \nu_3 = 4^2 31^2$ and $\nu_k = k431, k \geq 4$. Next, one can check with the help of modification rules

[9] To compare with [29] one has to take the eigenvalues of D of the form $e^{\pm\epsilon_1}, e^{\pm\epsilon_2 \pm\epsilon_3}$

that

$$sc_{\nu_k} + sc_{\lambda_{k,1}} + sc_{\lambda_{k,3}} + sc_{\lambda_{k,4}} = 0 \tag{4.35}$$

vanishes identically. Further, from fig. 12 each of the modules $\Delta_L(\lambda_{k,l}), l = 1, 3, 4$ has a single irreducible component $B_L(\nu_k)$. Finally, taking $L = |\nu_k|$ one can see from eq. (4.35) that the contribution of χ_{ν_k} to eq. (4.4) cancels out.

Let us introduce the compact notations $B_{k,l} := B_L(\lambda_{k,l})$ and $G_{k,l} := G(\lambda_{k,l})$. Then, putting together eq. (4.34) and fig. 12 we get from eq. (4.4) the following content of indecomposable modules $\mathcal{I}G_{k,l}$ appearing in eq. (4.2)

$$\begin{array}{ccccc}
B_{k,0} & B_{k,2} & B_{k,1} & B_{k,3} & B_{k,l+1} \\
 & & & & \\
 & G_{k,0} & G_{k,1} & G_{k,2} & G_{k,l} \\
G_{k,0} & G_{k,2} & G_{k,2} & G_{k,0}\ G_{k,1}\ G_{k,3} & G_{k,l-1}\ G_{k,l+1}, \\
 & G_{k,0} & G_{k,1} & G_{k,2} & G_{k,l}
\end{array} \tag{4.36}$$

where $l = 3, \ldots, m$ and $\lambda_{k,m} \vdash L$. The indecomposable modules $\mathcal{I}G_{k,l}$ are represented below $B_{k,l}$ and it should be understood that they get "paired up" in the decomposition of $V^{\otimes L}$ as a $\mathrm{OSp}(4|2) \times B_L(2)$ bimodule.[10] Alternative, maybe more intuitive physically, representations of the blocks will be given in the next paper.

The structure of modules $\mathcal{I}G_{k,l}$ is in perfect agreement with eq. (4.32,4.33). We recognize in the first term $\mathcal{I}G_{k,0} = G_{k,0}$ of eq. (4.36) the contribution to $V^{(0)}$, while the rest of the terms are exactly the projective modules appearing in $V^{(1)}$, that is $\mathcal{I}G_{k,2} = \mathcal{P}G_{k,0}, \mathcal{I}G_{k,1} = \mathcal{P}G_{k,1}$ and $\mathcal{I}G_{k,l} = \mathcal{P}G_{k,l-1}, l \geq 2$.

For typical $\lambda \in Y_L(1)$, the modules $\mathcal{I}G(\lambda) = G(\lambda)$ are irreducible and get paired up with $B_L(\lambda)$ in the decomposition of $V^{\otimes L}$ as a $\mathrm{OSp}(4|2)$-$B_L(2)$-bimodule.

Observe that, as expected, only the modules $B_{k,0}$ (which coincide with $B_L(\lambda_k)$ in eq. (4.11)) contribute to the supertrace $\mathrm{str}_{V^{\otimes L}}\, d$. Indeed, typical modules modules $G(\lambda)$ have superdimension 0. The same is true for projective modules. One can explicitly check from eq. (4.36) that $\mathrm{sdim}\,\mathcal{P}G_{k,l} = 0$ if we take into account that only osp(4|2) fermionic generators connect irreducible components of indecomposable modules. For instance, $\mathrm{sdim}\,\mathcal{P}G_{0,0} = \mathrm{sdim}\,G_{0,0} - \mathrm{sdim}\,G_{2,0} + \mathrm{sdim}\,G_{0,0} = 1 - 2 + 1 = 0$.

As we have explained at the beginning of sec. 4.1, the degeneracies of the eigenvalues of the OSp(4|2) spin transfer matrix are given by $\dim \mathcal{I}G(\lambda)$. We compute them in app. A.3.

Thus, in conclusion we see that $B_L(2)/J(1) = \mathrm{End}_{\mathbb{Z}_2 \times \mathrm{osp}(4|2)}\, V^{\otimes L}$ and, because $V^{\otimes L}$ is by definition a faithful $B_L(2)/J(1)$ module we also have $\mathbb{Z}_2 \times \mathrm{osp}(4|2) = \mathrm{End}_{B_L(2)/J(1)}\, V^{\otimes L}$. In other words, the two algebras $B_L(2)/J(1)$ and $\mathbb{Z}_2 \times \mathrm{osp}(4|2)$ are the full centralizers of each other on $V^{\otimes L}$.

This results allows us to relate the decomposition of $V^{\otimes L}$ as a OSp(4|2) left module to the decomposition of $V^{\otimes L}$ as a $B_L(2)$ right module.

Collecting in a single indecomposable module $\mathcal{I}B_{k,l}$ all factors $B_{k,l'}$ in eq. (4.36) which correspond to (happen to be above) an irreducible component $G_{k,l}$ we get [11]

$$\begin{array}{cccccc}
G_{k,0} & G_{k,1} & G_{k,2} & G_{k,l-1} & G_{k,m-1} & G_{k,m} \\
 & & & & & \\
B_{k,2} & B_{k,1} & B_{k,3} & B_{k,l} & B_{k,m} & \\
B_{k,0}\ B_{k,3} & B_{k,3} & B_{k,2}\ B_{k,1}\ B_{k,4} & B_{k,l-1}\ B_{k,l+1} & B_{k,m-1} & B_{k,m}, \\
B_{k,2} & B_{k,1} & B_{k,3} & B_{k,l} & B_{k,m} &
\end{array} \tag{4.37}$$

where $l = 4, \ldots, m-1$ and the content of $\mathcal{I}B_{k,l}$ is represented below $G_{k,l}$.

Apart the last irreducible module $\mathcal{I}B_{k,m} = B_{k,m}$, we recognize in the terms of eq. (4.37) the projective representations of the quiver E_∞ in fig. 12, which describes the homomorphisms between the $B_L(2)$ tensor modules $\delta_L(\lambda)$ realized on $V^{\otimes L}$.

---

[10] In pedantic terms, the pairing $B_{k,l}, \mathcal{I}G_{k,l}$ can be represented by the functor $B_{k,l} \to V^{\otimes L} \otimes_{B_L(2)} B_{k,l}$ sending $B_L(2)$ left modules to $\mathbb{Z}_2 \times \mathrm{osp}(4|2)$ left modules.

[11] Again this "collecting" can be represented by the functor $G_{k,l} \to \mathrm{Hom}_{\mathbb{Z}_2 \times \mathrm{osp}(4|2)}(V^{\otimes L}, G_{k,l})$ sending $\mathbb{Z}_2 \times \mathrm{osp}(4|2)$ left modules to $B_L(2)$ left modules.

5 The hamiltonian limit

It will turn out in our forthcoming analysis of conformal properties to be easier to study numerically the hamiltonian

$$H_\Delta = -\frac{1+\Delta}{2}\sum_{i=1}^{L-1}(I+P_i) - \frac{1-\Delta}{2}\sum_{i=1}^{L-1}E_i.$$

The expectation — which we will confirm in great details — is that this hamiltonian will be in the same universality class as the spin model we had started with.

The hamiltonian H_Δ is obviously local and has only nearest neighbour interactions if the E's and P's are taken in the spin representation provided by eq. (3.2). However, this is no longer true if we think of H_Δ as an element of the adjoint representation of $B_L(2)$.

The lowest eigenvalue of H_Δ belongs to the $B_L(2)$ irreducible representation labeled by $\mu = L \mod 2$.

For generic Δ it is nondegenerate if L is even and has degeneracy $\dim V = 4S+2$ if L is odd. On the other hand, the highest eigenvalue belongs to the completely antisymmetric representation labeled by $\mu = 1^L$. In this representation the P's act as -1 and the E's as 0.

The hamiltonian H_Δ is determined up to an arbitrary additive constant and multiplicative factor. For numerical diagonalization it is convenient to fix the additive constant such that the maximal eigenvalue of H_Δ be zero. The multiplicative factor is fixed by requiring

$$H_\Delta\Big|_{S=0} = H_{XXZ} + \text{cst}$$

with I, E, P as in eq. (3.2), J as in app. A.1 and H_{XXZ} being the XXZ spin chain hamiltonian in its usual form

$$H_{XXZ} = -\frac{1}{2}\sum_{i=1}^{L-1}\left(\sigma_i^x\otimes\sigma_{i+1}^x + \sigma_i^y\otimes\sigma_{i+1}^y + \Delta\,\sigma_i^z\otimes\sigma_{i+1}^z\right).$$

The fact that the eigenvalues of the 6 vertex model appear as a subset of the eigenvalues of the transfer matrix for the $\mathrm{OSp}(2S+2|2S)$ model and $S\geq 1$ carries over to a similar result for the hamiltonians. The velocity of sound for the massless excitations can thus be derived from its value for the XXZ subset, which is well known from [30] to be

$$v_s = \frac{\pi\sin\lambda}{\lambda},\quad \Delta = -\cos\lambda. \tag{5.1}$$

The hamiltonian H_Δ is diagonalized numerically in the adjoint representation of the Brauer algebra by studying its action on the diagrams just like for the transfer matrices. Next, once the structure of indecomposable modules $\Delta_L(\mu)$ is known, eq. (4.4) can be used as explained in sec. 4.1 to select the part of the spectrum which does indeed appear for a fixed S spin model.

However, in the two special cases $\Delta = \pm 1$ the hamiltonian H_Δ greatly simplifies. In the following two subsections we discuss the behaviour of the spectrum of H_Δ in the two limits $\Delta \to \pm 1^\mp$.

5.1 The limit $\Delta = 1$

When $\Delta = 1$ the Temperley Lieb operators E_i do not contribute to H_Δ and, thus, the hamiltonian is no longer a generic element of the Brauer algebra $B_L(2)$, but belongs instead to the subalgebra $\mathbb{C}\,\mathrm{Sym}(L)\subset B_L(2)$. This will translate to additional degeneracies in the spectrum of H_Δ at the point $\Delta = 1$ compared to other points in the range $-1\leq\Delta<1$.

Hamiltonians of type $-\sum P_i$, with P's in the representation provided by eq. (3.2), are integrable and have been studied in [31] and [32]. Although the continuum limit of such spin chains is a gapless field theory, it fails to be conformal, because excitations have a L^{-2} scaling law in the thermodynamic limit. This can readily be seen from the vanishing of the sound velocity in eq. (5.1). We will not enter into the details here, but just mention that the different systems of Bethe ansatz equations are indexed by $(2S+2, 2S)$-hook shape partitions $\lambda\vdash L$. This is exactly the label of irreducible representations of the group algebra $\mathbb{C}\,\mathrm{Sym}(L)$ realizing in the centralizer of the spin chain $V^{\otimes L}$, with V being the fundamental representation of $\mathrm{SU}(2S+2|2S)$. We see that the symmetry of our spin model $\mathrm{OSp}(2S+2|2S)$ jumps to $\mathrm{SU}(2S+2|2S)$ at the point $\Delta = 1$.

The additional degeneracies in the spectrum of the OSp$(2S+2|2S)$ spin model at the point $\Delta = 1$ can be understood by looking at the decomposition of $B_L(2)$ modules $\Delta_L(\mu)$, into a direct sum of $\mathbb{C}\,\mathrm{Sym}(L)$ irreducible modules $S(\lambda)$. Let $\mu \vdash L-2k$ and $\lambda \vdash L$, then it was shown in [33] that the multiplicity of $S(\lambda)$ in the decomposition of $\Delta_L(\mu)$ is

$$m(\mu,\lambda) = \sum_{\substack{\eta\vdash 2k\\ \eta \text{ even}}} c^{\lambda}_{\mu\eta}, \tag{5.2}$$

where $c^{\lambda}_{\mu\eta}$ are Littlewood-Richardson coefficients. Alternatively, $m(\mu,\lambda)$ is the number of tensors of rank $L-2k$, with index symmetry of some fixed standard Young supertableau of shape μ, that can be obtained from a tensor of rank L, with index symmetry of some standard Young supertableau of shape λ, by contracting $2k$ indices in all the possible ways.

One can apply eq. (5.2) to understand the degeneracy of the lowest level of H_Δ at $\Delta = 1$. First, observe that $-\sum P_i$ is minimized in the sector $\lambda = L$ (where P's acts as 1). The *only* μ such that $m(\mu,\lambda) \neq 0$ are one row partitions. Thus, the lowest eigenvalues of H_Δ restricted to $\Delta_L(L-2k)$ for $k = 0,\ldots,[L/2]$ become all degenerate at $\Delta = 1$.

Arguments of this kind can be used to derive information about the critical exponents of the spin model in the limit $\Delta \to 1^-$.

5.2 The limit $\Delta = -1$

The same reasoning can be applied to the point $\Delta = -1$. At this point, the hamiltonian H_Δ belongs to the Temperley Lieb subalgebra $T_L(1) \subset B_L(2)$ and the model can be considered as a spin chain $(V \otimes \bar{V})^{\otimes \frac{L}{2}}$ where $V, \bar{V}$ are the fundamental representation of SU$(2S+2|2S)$ and its conjugate. Additional degeneracies can be understood by looking at the decomposition of $B_L(2)$ modules $\Delta_L(\mu)$ as a direct sum of standard irreducible $T_L(1)$ modules $D_L(j)$.

Let us compute the multiplicity $n_L(\mu, j)$ of irreducible modules $D_L(j)$ in the decomposition of $\Delta_L(\mu)$ with $\mu \vdash L-2k$.

As explained in sec. 3.2, $\Delta_L(\mu)$ has a natural basis composed of all possible pairings $p \otimes v_i$ of partial diagrams p with $m = L-2k$ free points and basis vectors $v_1,\ldots,v_{f_\mu}$ of $S(\mu)$. We say that a horizontal line of a partial diagram p is *intersected* either if it intersects another horizontal line or if there is a free point in p between the two ends of the horizontal line. Let us associate to each partial diagram p the number of intersected horizontal lines l in p. It is not hard to see that the span on the basis vectors $p \otimes v_i$, with p's having at most l horizontal intersected lines, is a $T_L(1)$ submodule in $\Delta_L(\mu)$. If we denote this submodule by $\Delta^l_L(\mu)$ there is an obvious filtration $\Delta_L(\mu) = \Delta^k_L(\mu) \supset \cdots \supset \Delta^0_L(\mu) \supset \Delta^{-1}_L(\mu) = 0$ of $\Delta_L(\mu)$.

Consider the natural action of $T_L(1)$ on the quotient modules $Q^l_L(\mu) = \Delta^l_L(\mu)/\Delta^{l-1}_L(\mu)$. Observe that the action of $T_L(1)$ changes the labeling $\pi \in \mathrm{Sym}(m)$ of free points in a labeled graph $p \otimes \pi$ if and only if it also reduces the number of horizontal intersected lines. Therefore, $Q^l_L(\mu)$ is isomorphic to a direct sum of f_μ modules $Q^l_L(m)$. Obviously $Q^0_L(m) \simeq D_L(m)$ and, therefore, we get $n_L(\mu, j) = 0$ for $j < m$, $n_L(\mu, j) = f_\mu n_L(m, j)$ for $m \leq j \leq L$ and finally $n_L(m,m) = 1$.

Thus, our problem effectively reduces to understanding the action of $T_L(1)$ on the module $\Delta_L(m)$, which is composed of partial diagrams p on L points with m unlabeled free points.

At a closer look, one can see that the action of $T_L(1)$ on partial diagrams keeps the reciprocal configuration of intersected lines and free points intact. In other words, if ψ is a map that eliminates all the nonintersected horizontal lines from a partial diagram and acts as identity otherwise, then ψ defines an invariant of $T_L(1)$, that is

$$\psi(E_i \cdot p) = \psi(p), \quad i = 1,\ldots,L.$$

To understand the meaning of this invariant let us define a local map ϕ between partial diagrams which sends intersected horizontal lines to free points as depicted in fig. 13 and acts as identity otherwise. The local map ϕ is applied repeatedly until there are no more horizontal intersected lines left. It is not hard to see that ϕ extends to a homomorphism of $T_L(1)$ modules

$$\phi : Q^l_L(m) \to D_L(2l+m).$$

In fact, the role of the map ϕ is to show that $Q^l_L(m)$ is composed of a direct sum of isomorphic $D_L(2l+m)$ modules, while that of the map ψ is to distinguish between these modules. The set of partial diagrams p in

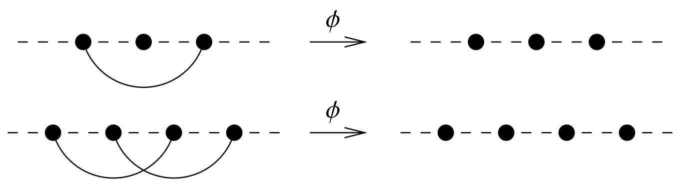


Figure 13: Illustration of the nontrivial local action of the map ϕ.

$Q^l_L(m)$ splits into subsets of constant $\psi(p)$ and each of these subsets is isomorphic to $D_L(2l+m)$ as a $T_L(1)$ module.

According to what was said before, we get that $n_L(m, j)$ equals to the number of graphs on $j = m + 2l$ vertices and l intersected edges. It follows that $n_L(m, j)$ does not actually depend on L and we drop the index L in the following. This fact allows, in principle, for an iterative computation of $n(m, j)$ by simply computing the dimensions of the left and right hand sides of the decomposition formula

$$\Delta_L(m) \simeq \bigoplus_{l=0}^{k} n(m, m+2l) D_L(m+2l),$$

that is

$$(2k-1)!!C^{2k}_L = \sum_{l=0}^{k} n(m, m+2l)\left(C^{k-l}_{L-1} - C^{k-l-2}_{L-1}\right) \tag{5.3}$$

successively for $L = 0, 2, \ldots$ or $L = 1, 3, \ldots$. One can give an explicit expression for $n(m, j)$ with a little more combinatorial work.

We call a horizontal line an *empty cup* if its ends are adjacent and simply a *cup* if there are separated by free points. Observe that all the lines in a partial diagram are intersected if and only if there is at least one free point in each cup. Thus, if the partial diagram has p cups with only one free point inside and a total of l edges then the remaining $m - p$ free points can be added to the diagram in C^{2l}_{m-p+2l} different ways in such a way that the resulting diagram has only intersected edges. Moreover, the number of diagrams on $2l$ points with p empty cups and a total of l edges is again $n(p, 2l - p)$. This is because the condition of no cups in the connection of the remaining $l - p$ edges is similar to the condition of composing a graph with p free points and $l - p$ intersected edges. Putting everything together we get a new recurrence formula

$$n(m, m+2l) = \sum_{p=0}^{l} C^{2l}_{m-p+2l} n(p, 2l-p) \tag{5.4}$$

reducing the problem to the computation of $n(j) := n(0, 2j)$.

Next, we want to find a recurrence relation for $n(j)$ by looking at the connectivity of the first point in the partial diagrams on $2j$ points with j intersected edges. The leftmost vertex in the partial diagram has to be connected to some other vertex at position k. The connectivity of the $2j - 1$ points to the left of the point at position 1 is equivalent to that in a partial diagram with $j - 1$ intersected edges and a free point except for the case where $k = 2$. Therefore we have that

$$n(j) = n(1, 2j-1) - n(j-1). \tag{5.5}$$

Now, eq. (5.4) yields $n(1, 2j-1) = (2j-1)n(j-1) + n(1, 2j-3)$. Using again eq. (5.5) for $j - 1$ we finally get that

$$n(j) = (2j-1)n(j-1) + n(j-2). \tag{5.6}$$

The solution of the recurrence eq. (5.6) with the initial conditions $n(1) = 0$ and $n(2) = 1$ is

$$n(j) = \sum_{k=0}^{j} (-1)^{j-k} \frac{(j+k)!}{2^k (j-k)!k!}$$

and coincides with the absolute value of Bessel polynomials $y_j(x)$

$$y_j(x) = \sum_{k=0}^{j} \frac{(j+k)!}{(j-k)!k!}\left(\frac{x}{2}\right)^k$$

evaluated at $x = -1$.

6 Conclusion

Besides the careful definition of the spin model and its sectors, the main point of this first paper is the algebraic set up necessary to analyze its symmetries. This is a non trivial task since we are dealing with non semi-simple algebras, and that the action of $\mathrm{OSp}(2S+2|2S)$ and $B_L(2)$ are meshed through a complex structure of indecomposable representations. The main results are the decomposition formulas (4.36,4.37) for $V_{4|2}^{\otimes L}$ viewed as a OSp(4|2) and a $B_L(2)$ module. The decomposition in eq. (4.36) has been computed in two essentially different ways: first, by decomposing tensor products between OSp(4|2) representations and V without knowing anything about the Brauer algebra and, second, starting from eq. (4.4) with the assumption that the representations of OSp(4|2) and $B_L(2)$ on $V^{\otimes L}$ generate the full centralizers of each other (Schur duality). The fact that we arrive at the same result using both methods highly suggests that our assumption about the Schur duality between OSp(4|2) and $B_L(2)$ on $V^{\otimes L}$ is correct.

When the question of decomposing $V^{\otimes L}$ is addressed in sec. 4.3, the notion of block appears to be a particularly useful concept for organizing indecomposable representations.[12] These results will be applied to educated conjectures about the conformal field theory in the next paper.

Although there are many things left unclear about the representation theory of $\mathrm{osp}(2S+2|2S)$, $S > 1$, it is very tempting to speculate the form of the decomposition of $V_{2S+2|2S}^{\otimes L}$. Before making the guess, observe that as a OSp(4|2) module $V_{4|2}^{\otimes L} \simeq T \oplus P$, where P is a direct sum of projectives organized in blocks, while T is a direct sum of simples indexed by the same Young tableau (in the partition notation for dominant weights) as the irreps of O(2). More than that, they appear with the same multiplicities as their partners in $V_{2|0}^{\otimes L}$.[13] Therefore, T and $V_{2|0}^{\otimes L}$ are similar in all but the internal structure of their simple summands. The similarity between the two modules has to be understood in terms of their centralizers, because these are precisely the objects that do not "see" the internal structure of simples.[14] In conclusion, one should have $\mathrm{End}_{\mathrm{O}(2)}\, V_{2|0}^{\otimes L} \simeq \mathrm{End}_{\mathrm{osp}(4|2)}\, T$, which is quite natural once there is a Schur duality between OSp(4|2) and $B_L(2)$ on $V_{2S+2|2S}^{\otimes L}$. It is tantalizing to speculate that as a $\mathrm{OSp}(2S+2|2S)$ module $V^{\otimes L} \simeq T \oplus P$, with P projective and $\mathrm{End}_{\mathrm{OSp}(2S+2|2S)}\, T \simeq \mathrm{End}_{\mathrm{OSp}(2S|2S-2)}\, V_{2S|2S-2}^{\otimes L} \simeq B_L(2)/J(S-1)$. Thus, the problem of the decomposition of $V_{2S+2|2S}^{\otimes L}$ as a $\mathrm{OSp}(2S+2|2S)$ module is reduced to understanding the projective representations of the supergroup, i.e. to finding the quiver diagram for each block. It has been suggested in [34] that the quiver diagram of blocks does not depend on S provided the degree of atypicality k and the action of the outer automorphism τ are fixed.[15] The discussion of sec. A.3 suggests that the two types of quivers for a block of $\mathrm{osp}(2S+2|2S)$ and a fixed k will give rise to the same quiver for the induced blocks in $\mathrm{OSp}(2S+2|2S)$.

We also succeeded in computing the multiplicity of Temperley Lieb representations in a standard $B_L(2)$-module $\Delta_L(\mu)$. Finally, we gave a combinatorial description of $B_L(N)$ blocks as the set of minimal partitions dressed by balanced removable border strips and have shown that there is a similar description for $\mathrm{osp}(R|2S)$ blocks.

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[12] Let us note that the blocks appear already in the representation theory of *simple* Lie algebras if infinite dimensional representations are allowed. They are precisely the orbits of the shifted action of the Weyl group on the weight lattice.

[13] In is not hard to prove employing the methods we used in this paper and the results of [29] for osp(3|2) that the same phenomenon occurs for $V_{3|2}^{\otimes L}$. In this case T is the trivial representation.

[14] By a corollary of the Schur lemma, if S is a simple module for the algebra A then $\mathrm{End}_A\, S \simeq \mathbb{C}$.

[15] τ can act in two ways: either leave invariant all the weights in the block or pairwise transform some of them.

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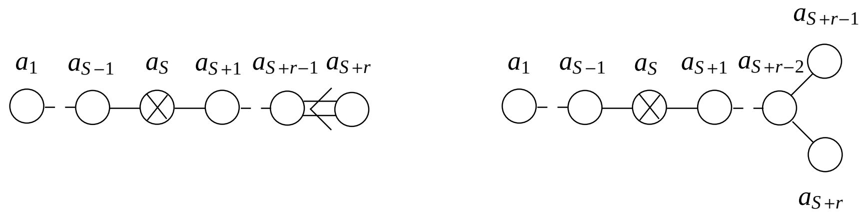

Figure 14: Distinguished Dynkin diagram for the Lie superalgebra $\mathrm{osp}(2r+1|2S)$ on the left and $\mathrm{osp}(2r|2S)$ on the right.

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A Appendix

A.1 $\mathrm{osp}(R|2S)$ Lie superalgebra

In this section we recall standard facts about the $\mathrm{osp}(R|2S)$ Lie superalgebra mainly following the pioneering work of Kac [35]. For more details on $\mathrm{osp}(R|2S)$ Young supertableaux see [36, 37, 38].

Let V be a vector space with an additive $\mathbb{Z}_2$ grading g, that is $V = V_0 \oplus V_1$ and $v \in V_\gamma \Rightarrow \mathrm{g}(v) = \gamma$. Let $\dim V_0 = R$, $\dim V_1 = 2S$ and $r = [R/2]$. Choose in V a basis $B = B_0 \cup B_1$ with $B_0 = \{v_i, v_i^* \in V_0, (v_{r+1} = v_{r+1}^*) \,|\, i = 1, \ldots, r\}$ and $B_1 = \{u_i, u_i^* \in V_1 \,|\, i = 1, \ldots, 2S\}$. We take the vector v_{r+1} in brackets because it appears for odd R only.

The grading of V induces a grading on $\mathrm{gl}(V, \mathbb{C})$, that is $\mathrm{gl}_0(V, \mathbb{C})$ preserves the degree of $v \in V_\gamma$ and $\mathrm{gl}_1(V, \mathbb{C})$ changes it. Define the supertranspose of a matrix $T \in \mathrm{gl}(V, \mathbb{C})$ by

$$T = \begin{pmatrix} A_{R\times R} & B_{R\times 2S} \\ C_{2S\times R} & D_{2S\times 2S} \end{pmatrix} \quad \Rightarrow \quad T^{\mathrm{st}} = \begin{pmatrix} A^{\mathrm{t}} & C^{\mathrm{t}} \\ -B^{\mathrm{t}} & D^{\mathrm{t}} \end{pmatrix}. \tag{A.1}$$

Let J denote the matrix with the only nonzero components

$$J_{vv^*} = 1, \quad J_{v*v} = 1, \quad J_{uu^*} = -1, \quad J_{u*u} = 1. \tag{A.2}$$

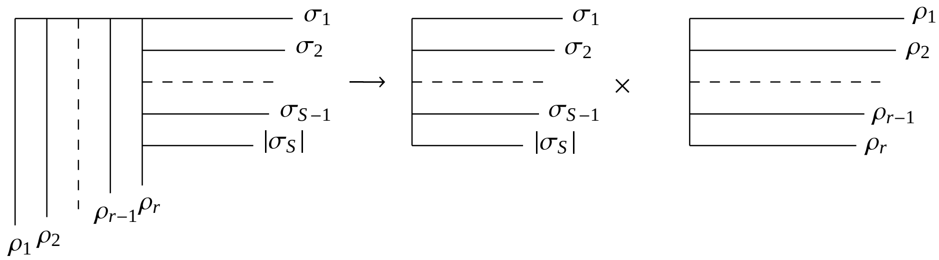

Figure 15: The so(R)×sp($2S$) representation to which belongs the highest weight state of a osp($R|2S$) representation.

The Lie superalgebra osp($R|2S$) is realized as a subset of gl($V,\mathbb{C}$) with elements T satisfying

$$T^{\rm st}J + JT = 0. \tag{A.3}$$

In terms of elementary matrices $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$ the generators of osp($R|2S$) read

$$T_{ij} = e_{ij} - e^{\rm st}_{i^*j^*} \tag{A.4}$$

$$T_{ij^*} = e_{ij^*} - (-1)^{{\rm g}(j)}e^{\rm st}_{i^*j} \tag{A.5}$$

$$T_{i^*j} = e_{i^*j} - (-1)^{{\rm g}(i)}e^{\rm st}_{ij^*}, \tag{A.6}$$

The generators $h_i = T_{ii}$ span the Cartan subalgebra $\mathcal{H}$. Denote by ε_i the basis in $\mathcal{H}^*$ dual to h_i. It can be easily checked that generators in eq. (A.4) correspond to roots of the type $\varepsilon_i - \varepsilon_j$, generators in eq. (A.5) correspond to roots of the type $\varepsilon_i + \varepsilon_j$ and generators in eq. (A.6) correspond to roots of the type $-\varepsilon_i - \varepsilon_j$. The bilinear invariant form $-\frac{1}{2}\,{\rm str}(h_ih_j)$ induces a scalar product on $\mathcal{H}^*$.

The standard basis is recovered by putting $\epsilon_i = \varepsilon_i$ for $i = 1,\ldots,r$ and $\delta_i = \varepsilon_{r+i}$ for $i = 1,\ldots,S$. Elementary weights δ_i, ϵ_j are orthogonal in $\mathcal{H}^*$ and $\delta_i^2 = -\epsilon_i^2 = 1$. The first $r+S-1$ simple roots are chosen to be $\alpha_i = \delta_i - \delta_{i+1}$, $\alpha_S = \delta_n - \epsilon_1$, $\alpha_{S+j} = \epsilon_j - \epsilon_{j+1}$ for $i = 1,\ldots,S$ and $j = 1,\ldots,r-1$. The last simple root is $\alpha_{r+S} = \epsilon_r$ for odd R and $\alpha_{r+S} = \epsilon_{r-1} + \epsilon_r$. The roots $\pm\delta_i \pm \epsilon_j$ are called odd and the rest — even.

The component of a weight Λ along the hidden simple sp($2S$) root $2\delta_S$ is

$$R \text{ odd}: \quad b = a_S - a_{S+1} - \cdots - a_{S+r-1} - a_{S+r}/2 \tag{A.7}$$

$$R \text{ even}: \quad b = a_S - a_{S+1} - \cdots - a_{S+r-2} - (a_{S+r-1} + a_{S+r})/2. \tag{A.8}$$

According to [35], an osp($R|2S$) highest weight is dominant iff it has integer Dinkyn labels $a_{i\neq S}$ and integer b satisfying the following consistency conditions

$$\begin{aligned} R \text{ odd}: &\quad b \le r-1 \Rightarrow a_{S+b+1} = \cdots = a_{S+r} = 0 \\ R \text{ even}: &\quad b \le r-2 \Rightarrow a_{S+b+1} = \cdots = a_{S+r} = 0, \quad b = r-1 \Rightarrow a_{S+r-1} = a_{S+r} = 0. \end{aligned} \tag{A.9}$$

All irreducible finite dimensional representations are indexed by dominant weights Λ. Given a dominant weight $\Lambda = \sum \rho_i\delta_i + \sum \sigma_j\epsilon_j$ in the standard basis, the first $r+S-1$ Dynkin labels are $a_i = \rho_i - \rho_{i+1}$ for $i = 1,\ldots,S$, $a_{S+i} = \sigma_i - \sigma_{i+1}$ for $i = 1,\ldots,r-1$. The last Dynkin label is $a_{S+r} = 2\sigma_r$ for R odd and $a_{S+r} = \sigma_{r-1} + \sigma_r$ for R even. From eq. (A.7) we also get $b = \rho_S$.

The set of numbers ρ_i, σ_j define a partition, shown in fig. 15, provided that consistency conditions (A.9) plus some additional constraints depending on R are satisfied. These additional constraints require $a_{S+r-1} < a_{S+r}$ and $a_{S+r-1} + a_{S+r}$ to be even if R is even, and a_{S+r} to be to be even if R is odd. The last two conditions define tensorial weights.

Partitions λ such that $\lambda_{r+1} \le S$ are called hook shape. Let τ denote the outer automorphism induced by the symmetry of the osp($2r|2S$) Dynkin diagram under the exchange of the last two roots in fig. 14. This automorphism is extremely important in understanding the difference between the representation theory of the supergroup OSp($R|2S$) and its Lie superalgebra. Note that τ can be explicitly realized through the discrete transformation ρ exchanging the last two basis vectors in B_0. Indeed, $\rho(\epsilon_j) = \epsilon_j$ for $j = 1,\ldots,r-1$ and $\rho(\epsilon_r) = -\epsilon_r$ because ϵ_j are the duals of $e_{jj} - e_{j^*j^*}$. Therefore $\rho(\alpha_{S+r}) = \rho(\epsilon_{r-1} + \epsilon_r) = \epsilon_{r-1} - \epsilon_r = \alpha_{S+r-1}$.

In the case of R odd, there is a bijective correspondence between hook shape partitions λ and dominant weights Λ. The same holds for R even, except for λ with $\sigma_r > 0$ when λ represents both Λ and $\tau \cdot \Lambda$.

If there is a pair (i, j) such that at least one of the conditions below are satisfied

$$\rho_j + \sigma_i + S + 1 - i - j = 0 \tag{A.10}$$

$$\rho_j - \sigma_i + S - R + 1 - j + i = 0, \tag{A.11}$$

the weight λ is called *atypical*.[16] See [39] for the origin of these conditions and note ref. [38], where these have been presented in the form (A.10,A.11). If none of these conditions is satisfied, the weight is called *typical* and, according to [39], the associated Kac module $\overline{V}(\Lambda)$ (which is a finite dimensional quotient of the corresponding highest weight module) is irreducible, its (super)character is given by the Weyl-Kac formula [40] and, in particular, its superdimension is zero.

A.2 OSp($R|2S$) supergroup

Let $\Gamma = \Gamma_0 \oplus \Gamma_1$ be a Grassman algebra. The supergroup OSp($R|2S$) may be realized as a subset of even supermatrices

$$M = \begin{pmatrix} A_{R\times R} & B_{R\times 2S} \\ C_{2S\times R} & D_{2S\times 2S} \end{pmatrix},$$

with entries in A and D belonging to Γ_0, and entries in B and C belonging to Γ_1, which satisfies

$$M^{\mathrm{st}} J M = J. \tag{A.12}$$

Equivalently, OSp($R|2S$) can be seen as the set of linear transformations leaving invariant the graded symmetric form

$$\eta_1.\eta_2 := \eta_1^{\mathrm{t}} J \eta_2 = \sum_{i=1}^{r} b_1^{i^*} b_2^{i} + b_1^{i} b_2^{i^*} + \left(b_{r+1}^2\right) + \sum_{j=1}^{S} f_1^{j^*} f_2^{j} - f_1^{j} f_2^{j^*}, \tag{A.13}$$

where η_α are arbitrary points in a superspace parametrized by coordinates $b_\alpha^i, b_\alpha^{i^*} \in \Gamma_0$ and $f_\alpha^j, f_\alpha^{j^*} \in \Gamma_1$ and $\alpha = 1, 2$.

Representing $M = I + \sum_a \alpha_a T_a$ with infinitesimal $\alpha_a \in \Gamma_0, \Gamma_1$ and expanding eq. (A.12) one gets the definition (A.3) of the superalgebra osp($R|2S$). Thus, the subgroup of OSp($R|2S$) connected to identity is an exponential of osp($R|2S$). The representation theory of both is the same as long as we restrict to tensor representations which are the only ones appearing in the tensor space $V^{\otimes L}$.

From the definition (A.12) any matrix $M \in \mathrm{OSp}(R|2S)$ has superdeterminant sdet $M = \pm 1$. The supergroup has two disconnected parts $\mathrm{OSp}^{\pm}(R|2S)$, which correspond to the value of the superdeterminant of its elements, that is $\mathrm{OSp}(R|2S)/\,\mathrm{OSp}^{+}(R|2S) = \mathbb{Z}_2$.

To see this, one can repeat the same reasoning typical of O(N) groups. Elementary transformations susceptible to change the sign of the superdeterminant belong to the discrete symmetry group W of the OSp($R|2S$) invariant form (A.13). The generators of W are read out from eq. (A.13) to be "reflections" $\rho_i : (b_i, b_i^*) \mapsto (b_i^*, b_i)$ and $\rho'_j : (f_j, f_j^*) \mapsto (-f_j^*, f_j)$, and permutations $\pi_i : (b_i, b_i^*) \leftrightarrow (b_{i+1}, b_{i+1}^*)$ and $\pi'_j : (f_i, f_i^*) \leftrightarrow (f_{i+1}, f_{i+1}^*)$. For odd R there is also the reflection $\rho_{r+1} : b_r \mapsto -b_r$. The subgroup W is in fact the Weyl group of the root system of $\mathrm{so}(R) \times \mathrm{sp}(2S)$. Denote by $W^{\pm}$ the set of elements of W embedded in $\mathrm{OSp}^{\pm}(R|2S)$. It is easy to see that all elements of W^- are conjugate in W^+ to a single reflection ρ, which one can take ρ_r if R is even and ρ_{r+1} if R is odd. Therefore, we see that indeed $W/W^+ = \mathbb{Z}_2$.

Let $v_{\Lambda'} \in g(\Lambda)$ be a vector of weight $\Lambda' \leq \Lambda$. Then, as seen in sec. A.1, there is an action of ρ on $g(\Lambda)$ provided by $\rho \cdot v_{\Lambda'} = v_{\tau\cdot\Lambda'}$. In the case of osp(4|2) the outer automorphism τ exchanges ϵ_2 with ϵ_3. The representations induced from osp($R|2S$) to OSp($R|2S$) are of the form

$$\mathrm{OSp}(R|2S) \otimes_{\mathrm{OSp}^{+}(R|2S)} g(\Lambda) \simeq \mathbb{Z}_2 \otimes_\rho g(\Lambda). \tag{A.14}$$

There are two possible cases now: i) either $\rho \cdot g(\Lambda) = g(\Lambda) \Leftrightarrow \tau \cdot \Lambda = \Lambda$ and then obviously $\mathbb{Z}_2 \otimes_\rho g(\Lambda) = 1 \otimes_\rho g(\Lambda) \bigoplus \varepsilon \otimes_\rho g(\Lambda)$ with $\rho \cdot 1 = 1$ and $\rho \cdot \varepsilon = -\varepsilon$ or ii) $\rho \cdot g(\Lambda) \neq g(\Lambda) \Leftrightarrow \tau \cdot \Lambda \neq \Lambda$ and the induced module in eq. (A.14) is irreducible.

[16] For $b \leq r - 1$ the highest weight Λ is always atypical.

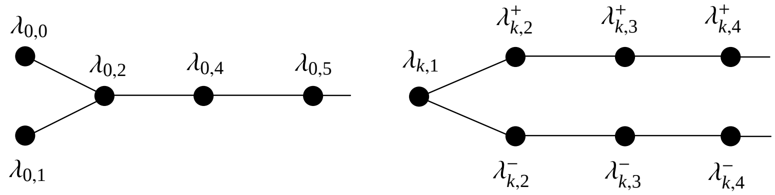

Figure 16: The quiver diagrams of type D_∞ and A_∞^∞ for the blocks of osp(4|2).

Two representations $R(\rho), R^*(\rho) = -R(\rho)$ of $\mathbb{Z}_2$ are called *associate*. The modules $G(1\times\lambda) := 1\otimes_\rho g(\Lambda)$ and $G(\varepsilon\times\lambda) := \varepsilon\otimes_\rho g(\Lambda)$ are also called associate. In contrast, $G(\tau\times\lambda) := \mathbb{Z}_2\otimes_\rho g(\Lambda) \simeq \mathbb{Z}_2\otimes_\rho g(\tau\cdot\Lambda)$ is isomorphic to its associate because there is an equivalence transformation between $R(\rho)$ and $R^*(\rho)$ through the change of sing of basis vectors in the subspace $\rho\otimes g(\Lambda)$. Therefore $G(\tau\times\lambda)$ is called selfassociate.

A direct implication following from the definitions of (self) associate modules is $\mathrm{sch}_\mu(D) = \mathrm{sdet}\, D\, \mathrm{sch}_{\mu^*}(D)$, where μ, μ^* are (self)associate weights of OSp($R|2S$). For a selfassociate weight μ this equality implies $\mathrm{sch}_\mu(D) = 0$ if $\mathrm{sdet}\, D = -1$.

Note that the centralizer of $B_L(N)$ on $V^{\otimes L}$ is the direct product algebra $\mathbb{Z}_2\times\mathrm{osp}(R|2S)$ rather then osp($R|2S$). This algebra has the same tensor irreducible representations as the supergroup OSp($R|2S$).

A.3 osp(4|2) Lie superalgebra and OSp(4|2) supergroup

This is a compact resumé of the results presented in [29] plus some additional remarks on the representation theory of OSp(4|2).

The superalgebra osp(4|2) has minor differences with respect to the general context of osp($R|2S$) superalgebras, because of the isomorphism $\mathrm{so}(4)\simeq\mathrm{sl}(2)\times\mathrm{sl}(2)$. The even part of the superalgebra is $\mathrm{so}(4)\times\mathrm{sp}(2)\simeq\mathrm{sl}(2)\times\mathrm{sl}(2)\times\mathrm{sl}(2)$. The odd part is a representation of the even part of dimension $2\times2\times2$.

The standard basis vectors $\{\epsilon_1,\epsilon_2,\epsilon_3\}$ of $\mathcal{H}^*$ are normalized as $\epsilon_1^2=-1$, $\epsilon_2^2=\epsilon_3^2=1/2$. The even and the odd positive root systems are $\Delta_0^+=\{2\epsilon_1,2\epsilon_2,2\epsilon_3\}$ and $\Delta_1^+=\{\epsilon_1\pm\epsilon_2\pm\epsilon_3\}$. The simple roots are traditionally chosen as $\alpha_1=\epsilon_1-\epsilon_2-\epsilon_3$, $\alpha_2=2\epsilon_2$, $\alpha_3=2\epsilon_3$. The hidden root will then be $2\epsilon_1=\alpha_1+\alpha_2+\alpha_3$.

Consistency conditions (A.9) for a dominant weight $\Lambda=b\epsilon_1+a_2\epsilon_2+a_3\epsilon_3$, require $b=0\Rightarrow a_2=a_3=0$ and $b=1\Rightarrow a_2=a_3$. We associate to Λ a hook shape partition λ with symplectic part $\rho_1=b$ and orthogonal part $\sigma_1=(a_2+a_3)/2, \sigma_2=|a_2-a_3|/2$. To make the correspondence $\Lambda\to\lambda$ bijective we mark the partition λ by $\mathrm{sgn}(\sigma_1-\sigma_2)$ when $\lambda_2>1$.[17]

Atypicality conditions (A.10,A.11) take the form

$$\begin{aligned}\rho_1+\sigma_1=0,\quad \rho_1+\sigma_2-1=0\\ \rho_1-\sigma_1-2=0,\quad \rho_1-\sigma_2-1=0.\end{aligned}$$

The solutions can be parametrized by two integers k and l. For $k=0$ these are $\lambda_{0,l}=l1^l, l\geq0$, while for $k>0$, $\lambda_{k,1}=k, \lambda_{k,l}=kl1^{l-2}, 2\leq l\leq k$ and $\lambda_{k,l}=l(k+1)1^{l-1}, k+1\leq l$.

Denote by $g(\lambda)$ the osp(4|2) simple modules. For λ typical $g(\lambda)\simeq\bar{V}(\lambda)$ and $\dim V(\lambda)$ can be computed by decomposing $\bar{V}(\lambda)$ into (at most 16) representations of $\mathrm{sl}(2)^{\times3}$

$$\dim g(\lambda)=16(b-1)(a_2+1)(a_3+1).$$

The dimensions of $g_{k,l}:=g(\lambda_{k,l})$ can be computed with the help of character formulas given in [29].

For $k=0$ we get $\dim g_{0,0}=\mathrm{sdim}\, g_{0,0}=1$, $\dim g_{0,1}=17, \mathrm{sdim}\, g_{0,1}=1$ and $\dim g_{0,l}=D_l^3-3D_l, \mathrm{sdim}\, g_{0,l}=2, l\geq2$, where $D_j=2j+1$.

For $k>0$ we get $\dim g_{k,1}=4k^2+2, \dim g^\pm_{k,l}=D_kD_{k-1}D_{l-1}-D_{l-1}^2D_{l-2}-2D_{l-1}D_{l-2}, 2\leq l\leq k$ and $\dim g^\pm_{k,l}=D_l^2D_{l-1}+2D_lD_{l-1}-D_kD_{k-1}D_l, l\geq k+1$, and $\mathrm{sdim}\, g_{k,0}=2, \mathrm{sdim}\, g^\pm_{k,l}=2, l\geq1$.

[17] Any $\mathrm{so}(4)=\mathrm{sl}(2)\oplus\mathrm{sl}(2)$ irreps can be written as a couple (j_1,j_2) of sl(2) irreps. The sign attached to λ distinguishes between (j_1,j_2) and (j_2,j_1) when $j_1\neq j_2$.

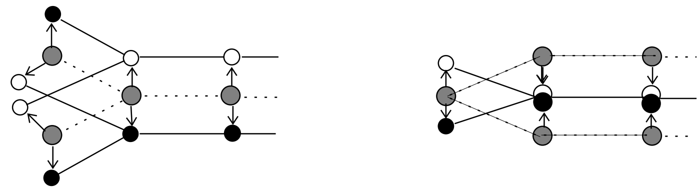

Figure 17: Schematic picture showing how the induction procedure (vertical arrows) sends the quiver diagrams of type D_∞ and A_∞^∞ (grey dots and dotted lines) for osp(4|2) blocks, represented in fig. 16, into quiver diagrams of type D_∞ for $\mathbb{Z}_2\times$osp(4|2) blocks. White and black dots represent weights of the type $1\times\lambda$ and $\varepsilon\times\lambda$ respectively. Double circles represent selfassociate weights $\tau\times\lambda$.

The set of weights $\lambda_{k,l}$ with k fixed belong to the same block of osp(4|2). To (at least partially) see this one has to check that the second order Casimir invariant takes the same value k^2 on the whole block k.[18] The actual construction of the set of indecomposable modules providing the equivalence relation of sec. B between the weights of a block is done in [29].

The quiver diagram representing the structure of osp(4|2) projective modules in a block is represented in fig. 16. The projective covers $\mathcal{P}g_{k,l}$ of the modules $g_{k,l}$ in the block $k=0$ have the submodule structure

$$\begin{array}{c} g_{0,0} \\ g_{0,2} \\ g_{0,0} \end{array} \qquad \begin{array}{c} g_{0,1} \\ g_{0,2} \\ g_{0,1} \end{array} \qquad \begin{array}{c} g_{0,2} \\ g_{0,0}\ g_{0,1}\ g_{0,3} \\ g_{0,2} \end{array} \qquad \begin{array}{c} g_{0,l} \\ g_{0,l-1}\ g_{0,l+1} \\ g_{0,l} \end{array} \qquad l\geq 3, \tag{A.15}$$

while in the block $k>0$ their submodule structure is

$$\begin{array}{c} g^-_{k,l} \\ g^-_{k,l-1}\ g^-_{k,l+1} \\ g^-_{k,l} \end{array} \qquad \begin{array}{c} g_{k,1} \\ g^-_{k,2}\ g^+_{k,2} \\ g_{k,1} \end{array} \qquad \begin{array}{c} g^+_{k,l} \\ g^+_{k,l-1}\ g^+_{k,l+1}, \\ g^+_{k,l} \end{array} \qquad l\geq 2. \tag{A.16}$$

The dimensions of projective modules in the block $k=0$ are $\dim\mathcal{P}g_{0,0}=112, \dim\mathcal{P}g_{0,l}=16(2l+1)(1+l+l^2), l\geq 1$, while in the block $k>0$ there are $\dim\mathcal{P}g_{k,1}=32(k^2-1), \dim\mathcal{P}g_{k,l}=16(2l-1)(k^2-1+l-l^2), l\leq k-1, \dim\mathcal{P}g_{k,k}=32(1+2k^2), \dim\mathcal{P}g_{k,l}=16(2l+1)(1-k^2+l+l^2), l\geq k+1$. The superdimension of projective (including typical) modules vanishes.

Let us apply the general discussion of sec. A.2 to the supergroup OSp(4|2). The outer automorphism τ acts on $\mathcal{H}^*$ by exchanging ϵ_2 with ϵ_3. Consequently, $\rho\cdot g_{k,l}=g_{k,l}$ for $k=0$ or $l=0$ and $\rho\cdot g^\pm_{k,l}=g^\mp_{k,l}$ otherwise.

We claim that the quiver diagram of type D_∞ for the block $k=0$ of osp(4|2) will give rise to two quiver diagrams of type D_∞, as shown in fig. 17, and, consequently, to two associate blocks for the algebra $\mathbb{Z}_2\times$osp(4|2), which we call $k=0,0^*$. As we shall see bellow, the weights in the block $k=0$ are $1\times\lambda_{0,0}, \varepsilon\times\lambda_{0,l}, l\geq 1$, while the weights in the block $k=0^*$ are $\varepsilon\times\lambda_{0,0}, 1\times\lambda_{0,l}, l\geq 1$.

We also claim that the quiver diagram of type A_∞^∞ for the block $k\neq 0$ of osp(4|2) will give rise to a single quiver diagram of type D_∞, as shown in fig. 17, and a selfassociate block for the algebra $\mathbb{Z}_2\times$ osp(4|2), which we label also by k. As we shall sea bellow, the weights in the block $k\neq 0,0^*$ are $1\times\lambda_{k,0}, \varepsilon\times\lambda_{k,0}, \tau\times\lambda_{k,l}, l\geq 1$.

Our claim follows from the analysis of shift operators $A^{\alpha\beta\gamma}$, $\alpha,\beta,\gamma=\pm$ introduced in [41]. These operators are very practical for decomposing Kac modules $\overline{V}(\lambda)$ into sl(2)$^{\times 3}$ irreps.[19] To be more specific, let $v_\mu\in\overline{V}(\lambda)$ be a vector of weight μ maximal for the algebra sl(2)$^{\times 3}$. Then, $A^{-\beta\gamma}v_\mu\neq 0$ is again a maximal vector of weight $\mu-\epsilon_1+\beta\epsilon_2+\gamma\epsilon_3$ for sl(2)$^{\times 3}$. To identify osp(4|2) irreducible components in $\overline{V}(\lambda)$ one has to search for sl(2)$^{\times 3}$ maximal vectors with dominant weights μ in the same block as λ.

[18]The second order Casimir is a central element of the enveloping superalgebra. The eigenvalues of central elements on $g(\lambda)$ define the *central character* of $g(\lambda)$. If two weights λ,λ' are in the same block then $g(\lambda), g(\lambda')$ have the same central characters. This is a consequence of the extension of the Schur lemma (in the form known to physicists) to indecomposable representations.

[19]The fact choose Kac modules in order to understand the transformation properties under ρ of arrows in the quiver diagram of a block is irrelevant for the following. One can take instead of $\overline{V}(\lambda)$ the standard modules $\mathcal{L}_0(\lambda)$ as well, which are defined by cohomological induction in [29].

Consider first the block $k = 0$ of osp(4|2). Then $\overline{V}(\lambda_{0,l}), l \geq 2$ has a $\mathrm{sl}(2)^{\times 3}$ maximal vector $A^{---}v_{\lambda_{0,l}}$ and one can check that all positive odd generators annihilate it. Therefore $A^{---}v_{\lambda_{0,l}}$ is a maximal vector for osp(4|2) and $\overline{V}(\lambda_{0,l})$ contains at least $g_{0,l}$ and $g_{0,l-1}$. In fact, these are the only two irreducible factors of $\overline{V}(\lambda_{0,l}), l \geq 3$ because, using the Weyl-Kac formula for characters and the results of sec. A.3, one can check that $\dim \overline{V}(\lambda_{0,l}) = \dim g_{0,l} + \dim g_{0,l-1}$. In the case $l = 2$ one has that $\dim \overline{V}(\lambda_{0,2}) - \dim g_{0,2} - \dim g_{0,1} = 1$ and, therefore, $\overline{V}(\lambda_{0,2})$ contains also the trivial representation.

In order to see how the four weights $1 \times \lambda_{0,l}, 1 \times \lambda_{0,l-1}, \varepsilon \times \lambda_{0,l}, \varepsilon \times \lambda_{0,l-1}$ split into two different blocks of $\mathbb{Z}_2 \times$ osp(4|2) one has to check out how A^{---} transforms under the action of ρ. From the explicit expression of shift operators in [41] it follows that $\rho A^{\alpha\beta\gamma}\rho = A^{\alpha\gamma\beta}$ and, consequently, $1 \times \lambda_{0,l}, 1 \times \lambda_{0,l-1}$ are in the same block and $\varepsilon \times \lambda_{0,l}, \varepsilon \times \lambda_{0,l-1}$ are in an other same block of $\mathbb{Z}_2 \times$ osp(4|2).

The module $\overline{V}(\lambda_{0,2})$ has a maximal vector $A^{---}A^{-+-}A^{--+}v_{\lambda_{0,2}}$, of weight zero, corresponding to the trivial representation $g_{0,0}$. With the help of relations in appendix [41] for the shift operator products of type $(1, 0, 0)$, one can show that $\rho A^{---}A^{-+-}A^{--+}v_{\lambda_{0,2}} = A^{--+}A^{-+-}A^{---}v_{\lambda_{0,2}} = -A^{-+-}A^{--+}A^{---}v_{\lambda_{0,2}}$ and, therefore, $g_{0,0}$ belongs to the block $k = 0$ of $\mathbb{Z}_2 \times$ osp(4|2) as claimed.

Consider now the block $k > 0$ of osp(4|2). Then $\overline{V}(\lambda^{\pm}_{k,l})$ will have a single osp(4|2) maximal vector (besides $v_{\lambda^{\pm}_{k,l}}$) corresponding to the irrep $g^{\pm}_{k,l-1}$ given by $A^{---}v_{\lambda^{\pm}_{k,l}}$ if $l > k + 1$, $A^{---}A^{-\mp\pm}v_{\lambda^{\pm}_{k,k+1}}$ if $l = k + 1$ and $A^{-\mp\pm}v_{\lambda^{\pm}_{k,l}}$ if $2 \leq l \leq k$. The induced module $\mathbb{Z}_2 \otimes_\rho \overline{V}(\lambda^{+}_{k,l})$, $l \geq 2$ will be the sum of $\overline{V}(\lambda^{\pm}_{k,l})$ glued together by the action of ρ. Finally, the induced module $\mathbb{Z}_2 \otimes_\rho \overline{V}(\lambda^{+}_{k,2})$ has two irreducible components $1 \otimes g_{k,0}$ and $\varepsilon \otimes g_{k,0}$ with $\mathbb{Z}_2 \times$ osp(4|2) maximal vectors $(1 \pm \rho) \otimes A^{--+}v_{\lambda^{+}_{k,2}}$.

B Blocks, minimality and atypicality

In this section we explain carefully the notion of block appearing in the representation theory of nonsemisimple algebras. We also look in details at the similarity between the blocks of osp(R|2S) and $B_L(N)$.

In the representation theory of non semisimple algebras the block is an essential notion. The *blocks* are conjugacy classes of irreps with respect to the equivalence relation $\equiv$ defined as follows. Let $\mathcal{I}$ be the category of indecomposable modules of the algebra. Write $S_1 \equiv S_2$ if there is an indecomposable module in $\mathcal{I}$ with simple summands S_1, S_2. Extend the relation $\equiv$ by transitivity in order to get an equivalence. In a semisimple algebra the notion of block is irrelevant because indecomposable representation are irreducible and the congruence $\equiv$ becomes an equality.

A relevant example is the Temperley Lieb algebra, with fugacity for loops N in its adjoint/diagrammatic representation. For generic values of N the algebra is semisimple and, thus, has only completely reducible representation. Restricting to subsets of planar diagrams, with the number of vertical lines fixed to m, and treating all the other diagrams as zero, we get all irreps, which are parametrized by m. However, at special points $N = 2\cos \pi r'/r''$ with coprime integers r', r'', the algebra becomes nonsemisimple, irreps labeled by m become reducible and m becomes a label for a whole block of the algebra, see [22].

The irreducible components $B_L(\lambda)$ of indecomposable modules $\Delta_L(\mu)$, when the Brauer algebra $B_L(N)$ is nonsemisimple, where first studied by mathematicians Hanlon *et al* in [14]. Recently Martin *et al* gave a complete description for the blocks of the Brauer algebra in [23].

We introduce the same notation as in [23] to formulate their block result for $B_L(N)$. If the box ϵ is in the row i and column j of the Young tableau of a partition μ, then its content is $c(\epsilon) = j - i$. Two boxes $\epsilon, \epsilon' \in \lambda$ are called balanced if $c(\epsilon) + c(\epsilon') = 1 - N$. For two partitions $\mu \subset \lambda$, the skew partition λ/μ is called balanced if it is composed of balanced pairs of boxes.

The necessary condition for $\Delta_L(\mu)$ to contain $B_L(\lambda)$ is: i) $\mu \subset \lambda$ and λ/μ is balanced; ii) If N is even and the boxes of content $1 - N/2, -N/2$ in λ/μ are configured as shown in case *a* fig. 18, then the number of columns in this configuration is even.

The given necessary criterion has the structure of a partial ordering. If $\mu \subset \lambda$ satisfy i) and ii) we write $\mu \leq \lambda$. The splitting of the set of weights X_L into posets with respect to $\leq$ gives the blocks of $B_L(N)$. As shown in [23], there is a unique minimal partition in a block, which can serve as a label.

A sufficient criterion for the module $\Delta_L(\mu)$ to contain $B_L(\lambda)$ was derived in [23] and requires λ to be the least weight $\lambda \geq \mu$.

We want to give a combinatorial description of the weights in a block. Consider the Young tableau of a partition λ in the block of the minimal partition μ. Let ϵ_1 (ϵ'_1) be the box with the highest (lowest) content in the

a *b*

Figure 18: Two possible configurations of boxes with content $1 - N/2, -N/2$.

skew partition λ/μ. Let ϵ_2 (ϵ'_2) denote the box bellow (on the left of) ϵ_1 (ϵ'_1), if there is one, and the box on the left of (above) ϵ_1 (ϵ'_1) otherwise. Define by recurrence the balanced pairs ϵ_i, ϵ'_i until $c(\epsilon_l) = -N/2 + 1, c(\epsilon'_l) = -N/2$ if N is even or $\epsilon_l = \epsilon'_l, c(\epsilon_l) = (1 - N)/2$ if N is odd. By construction, the set of boxes $\{\epsilon_i, \epsilon'_i\}_1^l$ belongs to a balanced removable border strip of width one or, simply, a *balanced strip*. One can repeat the same reasoning with the Young tableau of $\lambda/\{\epsilon_i, \epsilon'_i\}_1^l$ (which is not necessarily in the same block as μ because of ii)).

Thus, we clearly see that partitions λ in the same block can be constructed by dressing up with balanced strips a certain partition μ with no removable balanced strips. Denote by η the balanced strip of smallest length addable to μ. If N is odd denote by $\bar{\mu}$ the minimal partition $\bar{\mu}/\mu = \eta$. If N is even denote by $\bar{\mu}$ the minimal partition $\bar{\mu}/\mu = \eta$ only if the two boxes with content $-N/2, 1 - N/2$ in η are disposed horizontally and $\bar{\mu} = \mu$ otherwise. Partitions which are of the form μ dressed up with an even (odd) number of balanced strips are in the same block as μ ($\bar{\mu}$). Note that it is irrelevant in what order the strips are dressed on μ. Also, there cannot be two balanced strips of the same length. Thus, a partition λ in the block μ ($\bar{\mu}$) is unambiguously specified by the length of balanced strips in the skew partition λ/μ.

We claim now and show bellow that a block of $osp(R|2S)$ is composed, in the partition notation of sec. A.1, of hook shaped partitions built up by dressing with balanced strips an atypical partition with no removable balanced strips.[20] For that we need to reformulate the original block result [34] for $osp(R|2S)$.

Let the *degree of atypicality k* of a dominant weight Λ, be the dimension of the subspace $\mathcal{A}$ of the root lattice orthogonal to $\Lambda + \rho$, where ρ is the Weyl vector of $osp(R|2S)$. Each atypicality condition in eq. (A.10,A.11) is, in fact, an orthogonality condition between an odd root $\delta_i \pm \epsilon_j$, $\epsilon_j \neq 0$ and $\Lambda + \rho$. Therefore, k is the number of odd roots orthogonal to each other and to $\Lambda + \rho$ or, equivalently, the number of atypicality conditions labeled by couples (i, j) with distinct i and j. From the definition of the highest weight module $V(\Lambda)$ it is clear that irreducible finite dimensional components of $V(\Lambda)$ must have dominant weights of the form $\Lambda - \sum \mathbb{N}\alpha$, where the sum is over all odd positive roots α spanning $\mathcal{A}$.

Consider a $osp(R|2S)$ weight λ, which, in the notation of app. A.1, has symplectic part ρ and orthogonal part σ. Suppose ρ_{n+1}, ρ_{m+1} are the first columns of λ satisfying $\rho_{n+1} \leq r - S + n$ and $\rho_{m+1} \leq m + R - S - \rho_S - 1$. Then, one can find rows i_j, such that $\lambda_{i_j} < S$ and the pairs (i_j, j) satisfy the atypicality condition (A.11) for $m < j < n$ if R is odd and $m < j \leq n$ if R is even and the atypicality condition (A.10) for $n \leq j$. Indeed, from eq. (A.11) with $\sigma_i = 0$ the condition $m < j$ implies $i_j > \rho_S$ and thus $\lambda_{i_j} < S$, while $i_j \leq r$ implies $j \leq n$ if R is even and $j < n$ is R is odd. From eq. (A.10) with $\sigma_i = 0$ the condition $n \leq j$ implies $i_j \leq r$ while $i_j > \rho_S$ follows directly from $\rho_j \geq \rho_S$.

Conversely, if ρ_j, $m < j$ satisfies an atypicality condition with $\sigma_i = 0$, then $\lambda_i < S$. As shown in fig. 19, n is the width of the foot of the narrowest hook with arm width $r - S + n$ in which the Young tableau of λ can be drawn in.

Two atypicality conditions (i, j) and (i', j') are called independent if $i \neq i'$ and $j \neq j'$. Clearly, conditions (i_j, j) are pairwise independent for $m + 1 \leq j < n$ and for $n \leq j$. Let us show that an atypicality condition (i_j, j), $m + 1 \leq j < n$ is independent of conditions $(i'_{j'}, j')$, $n \leq j'$ iff there is a row shorter then S such that the box ϵ at the end this row and the box ϵ'_j at the end of column j are balanced.

In order to do that it is useful to imagine the partition λ drawn on an infinite square lattice, as in fig. 19, with each square having its content written inside. The following cases are possible

- Suppose first that there is no box with at the end of the row j. From $j < n$ follows $j \leq S - r$. Observe that the column $1 + S - r \leq j' = 1 + S - i_j$ is also empty. Therefore $i'_{j'} = i_j$ and the atypicality conditions (i_j, j) and $(i'_{j'}, j')$ are not independent.

[20]This means that condition ii) is relaxed when partitions are viewed as $osp(R|2S)$ weights. In particular the minimal partitions $\mu, \bar{\mu}$ discussed above are in the same block of $osp(R|2S)$ if there are hook shaped and atypical. Condition ii) is clearly related to the $\mathbb{Z}_2$ we neglect by looking at the representation theory of $osp(R|2S)$ instead of $OSp(R|2S)$.

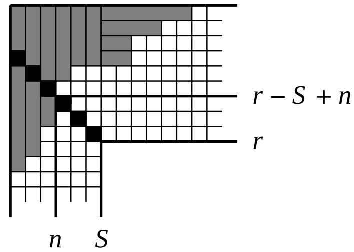

Figure 19: A partition λ drawn on a hook shaped square lattice and fitting exactly inside a hook with foot width n and arm width $r-S+n$. The black boxes represent the diagonal of squares with content $c = S-r$.

- Let $\rho_j \neq 0$ and let ϵ_j be the box at the end of column j. No suppose that λ has a rightmost box ω with content $c = 2-N-c(\epsilon'_j)$ and let j' be the column of that box. Condition $\rho_j \leq m+R-S-\rho_S-1$ gives $c \leq c(\epsilon'_S)+1+m-j$. The equality sign cannot hold because otherwise $j = j' = S$ which contradicts $j < n$. Thus $1+S-r \leq c = 1+S-i_j < c(\epsilon_S)$ and therefore $S-r \leq n \leq j' < S$. If $j'-\rho_{j'} < c$ then there is a box ϵ bellow ω, which is balanced with ϵ'_j and has no box to the right, thus it is the end of a row shorter then S. If $j'-\rho_{j'} = c$ then comparing the i's from the two atypicality conditions we get $i_j = i'_{j'}$ and the two atypicality conditions are not independent.
- Finally, if there is no box with content c then the column $j' = c$ gives an atypicality condition $(i'_{j'}, j')$ with $i_j = i'_{j'}$.

Next, by the definition of m, a column $j \leq m$ can satisfy an atypicality condition only with a row $\lambda_i \geq S$. After inserting $\sigma_i = \lambda_i - S$ in eqs. (A.10,A.11) we get

$$\lambda'_j + \lambda_i + 1 - i - j = 0 \tag{B.1}$$

$$(j-\lambda'_j) + (\lambda_i - i) = 1-N. \tag{B.2}$$

The lhs in eq. (B.1) is the hook length of the box in the row i and column j of λ, thus, always positive. On the other hand, eq. (B.2) requires the box ϵ'_j at the foot of column j be balanced with the box ϵ_i at the end of row i.

In the end, we see that there are two sources for *independent* atypicality conditions satisfied by a weight λ. First, if n is the width of the foot of the narrowest hook with arm width $r-S+n$, in which the Young tableau of λ can be drawn in, then there are $p := S-n$ atypicality conditions satisfied by the weight and we call them of type 1. Second, to each balanced pair of boxes ϵ, ϵ, such that ϵ is a box at the end a row and ϵ is a box at the end of a column, corresponds an atypicality condition of type 2. If λ satisfies q atypicality condition of type 2 then the degree of atypicality of the weight is $k = p+q$.

Let $\{\epsilon_{i_l}, \epsilon'_{j_l}\}_1^q$, $j_1 <, \ldots, < j_q$ be the set of balanced pairs satisfying atypicality conditions of type 2. Applying the iterative construction explained above to the boxes $\epsilon_{i_q}, \epsilon'_{j_q}$ one can see that there is a removable balanced strip η_{j_q} in $\lambda_q := \lambda$, with its ends in $\epsilon_{j_q}, \epsilon'_{j_q}$. Clearly, by the same reasoning, one can identify a new balanced strip η_{q-1} removable in $\lambda_{q-1} := \lambda_q/\eta_q$. The end λ_0 of this iterative procedure has no more removable balanced strips. Note that λ_0 satisfies k atypicality conditions all of type 1 and the sequence of weights $\lambda_0, \ldots, \lambda_q$ has the same degree of atypicality k.

In order to complete the proof of the claim it remains to notice two things. First, if α_i^1 is an odd root generating an atypicality condition of type 1, then $\Lambda - \sum_{i=1}^p \mathbb{N}\alpha_i^1$ is not dominant. Second, if $\alpha^2 = \delta_j + \epsilon_j$ is an odd root generating an atypicality condition of type 2, then Λ has a removable strip with its ends in the last box ϵ_i of row i and ϵ'_j of column j and $\Lambda - \alpha_i^2$ is dominant and can be represented by a partition of the form $\lambda/\{\epsilon_i, \epsilon'_j\}$, where λ is the Young tableau of Λ.

C Modification rules and $\mathrm{OSp}(R|2S)$ associate weights

The explicit form of the characters of classical groups is easier derived in the limit of infinite rank of the corresponding Lie algebra. The inverse limit exists and is given by the *modification rules* for characters. The concepts

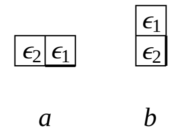

Figure 20: Configuration of lowest boxes ϵ_1, ϵ_2 in λ with content $1 - N/2, -N/2$ when N is even. Thick lines represent the border of the Young tableau of λ.

of infinite rank and inverse limit are rigorously defined for the case of Schur symmetric functions, connected to the irreducible characters of $\mathrm{GL}(N)$, in [42]. Let us clarify this point.

The characters of classical groups, evaluated on a group element, are polynomials in the eigenvalues of that element in the defining representation for the group. The infinite rank limit corresponds to considering polynomials depending on an infinite number of such variables. Irreducible characters are polynomials with a very specific symmetry, which is not obscured by the restriction of finite number of variables in the infinite rank limit.[21] These objects are known as *symmetric functions*. When the number of variables is set finite most symmetric functions become functionally dependent. Once an algebraically independent subset of symmetric functions is chosen, which is the actual set of characters in the case of classical groups, the modification rules "for characters" represent arbitrary symmetric functions along this basis.

One can introduce generalized symmetric functions sc_μ for the supergroup $\mathrm{OSp}(R|2S)$ according to eqs. (4.5, 4.6), see [25], [15]. The major difference with respect to classical groups is that functionally independent generalized symmetric functions are no longer irreducible characters of the supergroup. However, modification rules for sc_μ exist and have been derived in [26]. We bring them bellow in the form of eq. (4.8) with the notations of our paper.

Suppose that λ is a typical $\mathrm{osp}(R|2S)$ weight. Then, according to [26], only sc_μ with μ of the form λ dressed by balanced strips $\eta_1, \ldots, \eta_m$ modify to sc_λ

$$sc_\mu = \varepsilon^m w(\mu/\lambda) sc_\lambda, \tag{C.1}$$

here ε is the superdeterminant representation. We have also introduced the weight function $w(\mu/\lambda) = \prod_{i=1}^m (-1)^{c_i - 1}$ defined on skew partitions composed of balanced strips and c_i is the number of columns in η_i.

Consider now, the $\mathrm{osp}(R|2S)$ block labeled by the weight ν with degree of atypicality k and no removable balanced strips. Then, any weight λ in the block B_μ of μ is of the form ν dressed up by $q_\lambda \leq k$ balanced strips. Then, according to [26], sc_μ with μ of the form ν dressed up by $m \geq k+1$ balanced strips modifies to

$$sc_\mu = \sum_{\lambda \in \mathsf{B}_\nu} C_{m-k-1}^{m-q_\lambda-1} w(\mu/\lambda)(-1)^{k-q_\lambda} \varepsilon^{m-q_\lambda} sc_\lambda. \tag{C.2}$$

For a typical weight λ we put λ^* equal to λ dressed up by the balanced strip of minimal length η_1 if $\lambda'_S < r$. In order to prove that $sc_{\lambda^*} = \varepsilon sc_\lambda$ one has to show that η_1 runs over an odd number of columns.

Let us prove that η_1 runs over an odd number of columns c_1 if $\lambda'_S < r$ and an odd (even) number of columns if $\lambda'_S = r$ and R is odd (even).

Indeed, each box in η_1 belongs either to a horizontal or a vertical part of the strip, except for the boxes at the corners of η_1, which belong to both. We say the balanced pair $\epsilon, \epsilon' \in \eta_1$ has an allowable configuration if both boxes belong either to horizontal or vertical parts of η_1, otherwise ϵ, ϵ' has a non allowable configuration. As discussed in app. B, to every non allowable configurations of ϵ, ϵ' with content c, c' corresponds a removable balanced strip in λ with its ends in the border boxes with content $c-1, c'+1$ or $c+1, c'-1$ depending on weather ϵ is on a horizontal or a vertical part of η. Because λ is a $\mathrm{osp}(R|2S)$ typical weight and, thus, has no removable balanced strips, there are only allowable configuration of balanced pairs in η_1. Thus, a balanced pair in η_1, which is not in the same column, indexes either two different columns or none. There is at most one column containing the whole balanced pair and it appears always if N is odd and only for $\lambda'_S < r$ if N is even, as shown in fig. 20. Thus, c_1 is always odd for R odd and even only if $\lambda'_S = r$ for R even.

Let η_i denote the ith lowest length strip addable to λ and c_i the number of columns in it. One can prove by the same method that η_{i+1}/η_i has an even number of columns, one of which is already in η_i and, consequently, $c_{i+1} - c_i$ is odd.

[21] For instance, in the case of Schur symmetric functions this specific symmetry is the Littlewood-Richardson rule.

Again by the same method it is possible to prove that $\eta_1+\eta_2$ has an even number of columns. This is because η_2 contains a substrip η_1' which can be obtained by moving down along the diagonal the strip η_1. Applying what was said above about η_{i+1}/η_i to η_2/η_1' we see that $w(\eta_1+\eta_2) = -1$.

If λ is typical and $\lambda_S' < r$ then $\mu = \lambda + \eta_1 + \eta_2$ is the next partition in the block of λ, while if $\lambda_S' = r$ then $\mu = \lambda + \eta_1$ is the next partition in the bloc of λ. Therefore we have just shown, as claimed in sec. 4.3 that $sc_\mu = -sc_\lambda$.

Thus, the two cases in eq. (C.1) corresponding to the parity of m can be simply written as $sc_\mu = w(\mu/\lambda)sc_\lambda$ if $\lambda \preceq \mu$ and $sc_\mu = w(\mu/\lambda^*)sc_{\lambda*}$ if $\lambda^* \preceq \mu$ because $w(\mu/\lambda) = w(\mu/\lambda^*)$.

We do not now how to explicitly define the associates of atypical weights for general $\mathrm{osp}(R|2S)$.